A NEW PROOF OF THE BEST BOUNDS IN WALLIS' INEQUALITY

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Abstract. By using some properties of gamma function and psi function and the convolution theorem, a new proof of the following double inequality is given: For all natural number n, we have

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},$$

and the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

1. Introduction

Define $(2m)!! = \prod_{i=1}^{m}(2i)$ and $(2m-1)!! = \prod_{i=1}^{m}(2i-1)$ for any given positive integer m. Then we have

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{1}$$

The inequality (1) is called Wallis' inequality in [7, p. 103] and can be improved to the following

Theorem 1. *For all natural number n, we have*

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{2}$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

In [2, pp. 358–359] and [9], it was twice proved that the function $\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 - x$ is decreasing for $x>0$. This implies that the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ in the lower and upper bounds of inequality (2) are the best possible.

Recently, inequality (2) in Theorem 1 was obtained using different approaches by the authors in [3, 4, 5].

In this short note, we will give a new proof of Theorem 1 by using some properties of gamma and psi functions and the convolution theorem.

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2. Lemmas

The following lemmas regarding to gamma function $\Gamma(x)$ and psi function $\psi = \frac{\Gamma'}{\Gamma}$ are necessary.

Lemma 1 ([6]). *For $x > 0$, we have*

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\bigl(x^{-2}\bigr). \tag{3}$$

Lemma 2 ([1, 8]). *For $x > 0$, we have*

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\,\mathrm{d}t, \tag{4}$$

$$\psi(x) = \ln x - \frac{1}{2x} - \sum_{r=1}^{n} \frac{(-1)^{r-1}B_r}{2r} x^{-2r} + O\bigl(x^{-2n-2}\bigr), \tag{5}$$

where $\gamma = 0.57721566490153286060651\cdots$ is the Euler's constant. In particular,

$$\psi(x) = \ln x - \frac{1}{2x} + O\bigl(x^{-2}\bigr). \tag{6}$$

Lemma 3. *Let $f_1(t)$ and $f_2(t)$ be piecewise continuous for $t \ge 0$ on any given finite interval and there exist two constants $M > 0$ and $c \ge 0$ such that $|f(t)| \le Me^{ct}$, then we have*

$$\int_0^\infty \left[\int_0^s f_1(u) f_2(t-u)\,\mathrm{d}u\right] e^{-st}\,\mathrm{d}t = \int_0^\infty f_1(u)e^{-su}\,\mathrm{d}u \int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}v. \tag{7}$$

Remark 1. Lemma 3 is a convolution theorem of Laplace transform, which can be found in standard textbooks, for example, [1, 10].

3. A new proof of Theorem 1

Since

$$\Gamma(n+1) = n!, \quad \Gamma\Bigl(n + \frac{1}{2}\Bigr) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\,, \quad 2^n n! = (2n)!!, \tag{8}$$

the double inequality (2) can be rewritten as

$$\frac{1}{4} < \left[\frac{\Gamma(n+1)}{\Gamma\bigl(n+\frac{1}{2}\bigr)}\right]^2 - n \le \frac{4}{\pi} - 1. \tag{9}$$

Let

$$f(x) = \left[\frac{\Gamma(x+1)}{\Gamma\bigl(x+\frac{1}{2}\bigr)}\right]^2 - x, \quad x > 0. \tag{10}$$

Direct computation gives

$$f'(x) = 2\left[\frac{\Gamma(x+1)}{\Gamma\bigl(x+\frac{1}{2}\bigr)}\right]^2 \left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right)\right] - 1 \tag{11}$$

and

$$\begin{aligned} &\frac{\psi(x+1) - \psi\bigl(x+\frac{1}{2}\bigr)}{1 + f'(x)} f''(x) \\ &= \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + 2\left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right)\right]^2 \\ &\triangleq g(x). \end{aligned} \tag{12}$$

Differentiating (4) yields

$$\psi'(x) = \int_0^\infty \frac{te^{-xt}}{1-e^{-t}}\,\mathrm{d}t. \tag{13}$$

From (4) and (13), it follows that

$$g(x) = -\int_0^\infty te^{-xt}h(t)\,\mathrm{d}t + 2\left(\int_0^\infty e^{-xt}h(t)\,\mathrm{d}t\right)^2, \tag{14}$$

where

$$h(x) = \left(e^{t/2}+1\right)^{-1}. \tag{15}$$

By using the convolution theorem, Lemma 3, we have

$$\begin{aligned} g(x) &= -\int_0^\infty te^{-xt}h(t)\,\mathrm{d}t + 2\int_0^\infty \left[\int_0^t h(s)h(t-s)\,\mathrm{d}s\right]\mathrm{d}t \\ &= \int_0^\infty e^{-xt}I(t)\,\mathrm{d}t, \end{aligned} \tag{16}$$

where

$$I(t) = \int_0^\infty \left[2h(s)h(t-s) - h(t)\right]\mathrm{d}s. \tag{17}$$

We claim that for $0 < s < t$ the following inequality holds:

$$2h(s)h(t-s) - h(t) > 0, \tag{18}$$

which is equivalent to

$$\left(1+e^{s/2}\right)\left(1+e^{(t-s)/2}\right) < 2\left(1+e^{t/2}\right). \tag{19}$$

Let

$$J(t) = \ln\left(1+e^{s/2}\right) + \ln\left(1+e^{(t-s)/2}\right) - \ln\left[2\left(1+e^{t/2}\right)\right], \quad 0 < s < t.$$

Calculating straightforwardly yields

$$J'(t) = \frac{e^{t/2}\left[1-e^{s/2}\right]}{2e^{s/2}\left(1+e^{t/2}\right)\left(1+e^{(t-s)/2}\right)} < 0.$$

Therefore we have $J(t) < J(s) = 0$, which means that inequality (18) is valid.

Combining (16), (17) and (18) leads to $g(x) > 0$. From (13), it follows that $\psi'(x) > 0$, and $\psi(x)$ is increasing in $(0, \infty)$. Since $1 + f'(x) \geq 0$ by (11), $f''(x)$ and $g(x)$ have the same sign by (12), thus $f''(x) > 0$ and $f'(x)$ is increasing in $(0, \infty)$.

From (3), we have

$$\lim_{x\to\infty} x^{-\frac{1}{2}} \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} = 1, \tag{20}$$

From (6), it follows that

$$\lim_{x\to\infty} x\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] = \frac{1}{2}. \tag{21}$$

Combination of (11), (20) and (21) yields

$$f'(x) < \lim_{x\to\infty} f'(x) = 0,$$

which implies that $f(x)$ is decreasing in $(0, \infty)$. Hence

$$\lim_{n\to\infty}\left\{\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n\right\} < \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n \leq \left[\frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2})}\right]^2 - 1 = \frac{4}{\pi} - 1. \tag{22}$$

We can rewrite $f(x)$ as

$$f(x)=x\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}-1\right]\left[x^{-1/2}\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}+1\right]. \tag{23}$$

Using (3) yields

$$\lim_{n\to\infty}\left\{\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2-n\right\}=\lim_{x\to\infty}f(x)=\frac{1}{4}. \tag{24}$$

The double inequality (2) follows from (22) and (24), and the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. The proof is complete.

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