ON NONLINEAR INTEGRAL INEQUALITIES OF GRONWALL TYPE IN TWO VARIABLES

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Abstract. In this paper we obtain some new nonlinear integral inequality of Gronwall type involving functions of two independent variables which can be used in the analysis of the behavior of the solutions of some partial differential equations.

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1. Introduction

Closely related to the foregoing first-order ordinary differential operators is the following result of Bellman [4]: *If the functions $g(t)$ and $u(t)$ are nonnegative for $t \geq 0$, and if $c \geq 0$, then the inequality*

$$u(t) \leq c + \int_0^t g(s)u(s)\,ds, \quad t \geq 0,$$

implies that

$$u(t) \leq c\exp\left(\int_0^t g(s)\,ds\right), \quad \textit{for} \quad t \geq 0.$$

This result may be established either directly or by means of the technique of first-order linear differential equations (please, see Gronwall [8] and Guiliano [9]). Various applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Numerous applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [13], Bihari [5], and Langenhop [10]. Several authors generalized inequalities of Bellman type (sometimes, inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of Gronwall type") to the case of functions of two or more variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations. In the book by Beckenbach and Bellman [2] the following unpublished Wendroff result was given: *If*

$$u(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y v(r,s)u(r,s)\,drds, \tag{1.1}$$

where $a(x), b(y) > 0, a'(x), b'(y) \geq 0, u(x,y), v(x,y) \geq 0$, then

$$u(x,y) \leq \frac{(a(0)+b(y))(a(x)+b(0))}{a(0)+b(0)} \exp\left(\int_0^x \int_0^y v(r,s)\,drds\right).$$

The Wendroff inequality (1.1) was generalized by Bainov and Simeonov [1]: *Let $u(x,y), a(x,y), k(x,y)$ be nonnegative continuous functions for $x \geq x_0, y \geq y_0$, and let $a(x,y)$ be nondecreasing in each of the variables for $x \geq x_0, y \geq y_0$. Suppose that*

$$u(x,y) \leq a(x,y) + \int_{x_0}^x \int_{y_0}^y k(s,t)u(s,t)\,dtds, \quad x \geq x_0, y \geq y_0. \tag{1.2}$$

Then

$$u(x,y) \le a(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds\right), \quad x \ge x_0, y \ge y_0.$$

In a recent paper [14] Pachpatte has given some useful integral inequalities involving functions of two independent variables and presented some of its applications. Our main objective here is to obtain a bound on the nonlinear version of (1.2) and also establish some new nonlinear integral inequalities involving functions of two independent variables which can be used in the analysis of the behavior of the solutions of some terminal value problem for the hyperbolic partial differential equation.

2. Results

In this section we state and prove some new nonlinear integral inequalities in two independent variables. Throughout the paper, all the functions which appear in the inequalities are assumed to be realvalued and all the integrals are involved in existence on the domains of their definitions. We shall introduce some notation: R denotes the set of real numbers and $R_+ = [0,\infty)$, $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ are the given subsets of R. The first order partial derivatives of a functions $z(x,y)$ defined for $x, y \in R$ with respect to x and y are denoted by $z_x(x,y)$ and $z_y(x,y)$ respectively.

Theorem 2.1. *Let $u(x,y), a(x,y), k(x,y)$ be nonnegative continuous functions for $x \ge x_0, y \ge y_0$, and let $a(x,y)$ be nondecreasing in each of the variables for $x \ge x_0, y \ge y_0$. Suppose that*

$$u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u^p(s,t)\,dtds, \quad x \ge x_0, y \ge y_0, \tag{2.1}$$

where $p \ge 0, p \ne 1$, is a constants. Then

$$u(x,y) \le \left[a^q(x,y) + q\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds\right]^{1/q} \tag{2.2}$$

for $x \in [x_0, X), y \in [y_0, Y)$, where $q = 1 - p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[x_0, X)$ and $[y_0, Y)$.

Proof. Let $X > x_0$ and $Y > y_0$ be fixed. Then for $x_0 \le x \le X, y_0 \le y \le Y$ we have

$$u(x,y) \le a(X,Y) + \int_{x_0}^{x}\left(\int_{y_0}^{y} k(s,t)u^p(s,t)\,dt\right)ds. \tag{2.3}$$

Define a function $v(x,y)$ by the right-hand side of (2.3). Then the function $v(x,y)$ is nondecreasing in each variable x,y, and $v(x_0,y) = a(X,Y)$,

$$\frac{\partial v}{\partial x}(x,y) = \int_{y_0}^{y} k(x,t)u^p(x,t)\,dt \le \int_{y_0}^{y} k(x,t)\,dt v^p(x,y), \tag{2.4}$$

since $u(x,t) \le v(x,t) \le v(x,y)$. According to (2.4), the function $z(x,y) = v^q(x,y)/q$ satisfies

$$\frac{\partial z}{\partial x}(x,y) = v^{q-1}(x,y)\frac{\partial v}{\partial x}(x,y) \le \int_{y_0}^{y} k(x,t)\,dt. \tag{2.5}$$

Integrating (2.5) over s from x_0 to x, and the change of variable yields

$$z(x,y) \le \frac{1}{q}v^q(x_0,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds,$$

or

$$v^q(x,y) \lesseqgtr a^q(X,Y) + q\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds,$$

where $\le$ (respectively, $\ge$) holds for $q > 0$ (respectively, $q < 0$). In both cases this estimate implies

$$v(x,y) \le \left[a^q(X,Y) + q\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds\right]^{1/q}$$

for $x_0 \le x \le X, y_0 \le y \le Y$. Setting $x = X$ and $y = Y$ and changing notation we arrive at (2.2). □

Corollary 2.1. *Let $u(x,y), k(x,y)$ be nonnegative continuous functions for $x \ge x_0, y \ge y_0$, and let $a(x)$ be nondecreasing in $x, x \ge x_0$, and $b(y)$ be nondecreasing in $y, y \ge y_0$. Suppose that*

$$u(x,y) \le a(x) + b(y) + \int_0^x\int_y^{\infty} k(s,t)u^p(s,t)\,dtds, \quad x \ge x_0, y \ge y_0,$$

where $p \ge 0, p \ne 1$, is a constants. Then

$$u(x,y) \le \left[(a(x)+b(y))^q + q\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds\right]^{1/q}$$

for $x \in [x_0,X), y \in [y_0,Y)$, where $q = 1-p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[x_0,X)$ and $[y_0,Y)$.

Theorem 2.2. *Let $u(x,y), a(x,y), k(x,y)$ be nonnegative continuous functions in R^2_+, and let $a(x,y)$ be nonincreasing in each of the variables x, y. Suppose that*

$$u(x,y) \leq a(x,y) + \int_x^\infty \int_y^\infty k(s,t)u^p(s,t)\,dtds, \quad x \geq 0, y \geq 0,$$

where $p \geq 0, p \neq 1$, is a constants and

$$\int_x^\infty \int_y^\infty k(s,t)\,dtds < \infty, \quad x \geq 0, y \geq 0.$$

Then

$$u(x,y) \leq \left[a^q(x,y) + q\int_x^\infty \int_y^\infty k(s,t)\,dtds\right]^{1/q}$$

for $x \in [0,X), y \in [0,Y)$, where $q = 1-p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[0,X)$ and $[0,Y)$.

Proof. The details of the proof of Theorem 2.2 follows by an argument similar to that in the proofs of Theorem 2.1 with suitable changes. We omit the details. □

By a reasoning similar to the proof of Theorem 2.1 we also can prove the following assertions.

Theorem 2.3. *Let $u(x,y), a(x,y), k(x,y)$ be nonnegative continuous functions in R^2_+, and let $a(x,y)$ be nondecreasing in x and nonincreasing in y. Suppose that*

$$u(x,y) \leq a(x,y) + \int_0^x \int_y^\infty k(s,t)u^p(s,t)\,dtds, \quad x \geq 0, y \geq 0,$$

where $p \geq 0, p \neq 1$, is a constants and

$$\int_0^x \int_y^\infty k(s,t)\,dtds < \infty, \quad x \geq 0, y \geq 0.$$

Then

$$u(x,y) \leq \left[a^q(x,y) + q\int_0^x \int_y^\infty k(s,t)\,dtds\right]^{1/q}$$

for $x \in [0,X), y \in [0,Y)$, where $q = 1-p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[0,X)$ and $[0,Y)$.

Our next theorems deal with some generalizations of Theorem 2.1, Theorem 2.2 and Theorem 2.3.

Theorem 2.4. *Let $u(x,y), a(x,y), b(x,y), k(x,y)$ be nonnegative continuous functions for $x \geq x_0, y \geq y_0$, and let $a(x,y)$ be nondecreasing in each of the variables for $x \geq x_0, y \geq y_0$. Suppose that*

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x} b(s,y)u(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u^p(s,t)\,dtds \tag{2.6}$$

for $x \geq x_0, y \geq y_0$, where $p \geq 0, p \neq 1$, is a constants. Then

$$\begin{aligned} u(x,y) \leq &\exp\left(\int_{x_0}^{x} b(\tau,y)\,d\tau\right) \\ &\times \left[a^q(x,y) + q\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\exp\left(\int_{x_0}^{s} b(\tau,y)\,d\tau\right) dtds\right]^{1/q} \end{aligned} \tag{2.7}$$

for $x \in [x_0, X), y \in [y_0, Y)$, where $q = 1-p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[x_0, X)$ and $[y_0, Y)$.

Proof. Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u^p(s,t)\,dtds.$$

Then $z(x,y)$ is nondecreasing in each variables x, y, and (2.6) can be restated as

$$u(x,y) \leq z(x,y) + \int_{x_0}^{x} b(s,y)u(s,y)\,ds. \tag{2.8}$$

Further define a function $v(x,y)$ by $v(x,y) = \int_{x_0}^{x} b(s,y)u(s,y)\,ds$. Then $v(x_0,y) = 0$, we have

$$\frac{\partial v}{\partial x}(x,y) \leq b(x,y)z(x,y) + b(x,y)v(x,y), \tag{2.9}$$

since $u(x,y) \leq z(x,y) + v(x,y)$. The inequality (2.9) imply that

$$\left[\frac{\partial v}{\partial s}(s,y) - (s,y)v(s,y)\right]\exp\left(\int_{s}^{x} b(\tau,y)\,d\tau\right) \leq b(s,y)z(s,y)\exp\left(\int_{s}^{x} b(\tau,y)\,d\tau\right)$$

for $s \geq x_0$, or

$$\frac{\partial}{\partial s}\left[v(s,y)\exp\left(\int_{s}^{x} b(\tau,y)\,d\tau\right)\right] \leq b(s,y)z(s,y)\exp\left(\int_{s}^{x} b(\tau,y)\,d\tau\right).$$

Integration over s from x_0 to x gives

$$v(x,y) \le \int_{x_0}^{x} b(s,y)z(s,y)\exp\left(\int_{s}^{x} b(\tau,y)\,d\tau\right) ds,$$

which implies

$$v(x,y) \le z(x,y)\int_{x_0}^{x} b(s,y)\exp\left(\int_{s}^{x} b(\tau,y)\,d\tau\right) ds, \tag{2.10}$$

since $v(x_0,y)=0$. From (2.8) and (2.10), we get

$$u(x,y) \le z(x,y)\exp\left(\int_{x_0}^{x} b(\tau,y)\,d\tau\right). \tag{2.11}$$

Using the definition of $z(x,y)$ and (2.11) we find the estimate

$$z(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\exp\left(p\int_{x_0}^{s} b(\tau,t)\,d\tau\right) z^p(s,t)\,dt\,ds.$$

Now Theorem 2.1 implies

$$z(x,y) \le \left[a^q(x,y) + q\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\exp\left(p\int_{x_0}^{s} b(\tau,t)\,d\tau\right) dt\,ds\right]^{1/q}, \tag{2.12}$$

for $x \in [x_0,X), y \in [y_0,Y)$, where $q = 1-p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[x_0,X)$ and $[y_0,Y)$. The desired inequality in (2.7) follows by using (2.12) in (2.11). $\square$

Theorem 2.5. *Let $u(x,y), a(x,y), b(x,y), k(x,y)$ be nonnegative continuous functions in R_+^2, and let $a(x,y)$ be nonincreasing in each of the variables for x,y. Suppose that*

$$u(x,y) \le a(x,y) + \int_{x}^{\infty} b(s,y)u(s,y)\,ds + \int_{x}^{\infty}\int_{y}^{\infty} k(s,t)u^p(s,t)\,dtds$$

for $x \ge 0, y \ge 0$, where $p \ge 0, p \ne 1$, is a constants, and

$$\int_{x}^{\infty} b(s,y)\,ds < \infty, \qquad \int_{x}^{\infty}\int_{y}^{\infty} k(s,t)\,dtds < \infty$$

for $x \geq 0, y \geq 0$. *Then*

$$u(x,y) \leq \exp\left(\int_x^\infty b(\tau,y)\,d\tau\right) \times \left[a^q(x,y) + q\int_x^\infty \int_y^\infty k(s,t)\exp\left(\int_s^\infty b(\tau,y)\,d\tau\right) dtds\right]^{1/q}$$

for $x \in [0,X), y \in [0,Y)$, *where* $q = 1-p$, X *and* Y *are chosen so that the expression between [...] is positive in the subintervals* $[0,X)$ *and* $[0,Y)$.

Proof. The details of the proof of Theorem 2.5 follows by an argument similar to that in the proofs of Theorem 2.4 with suitable changes. We omit the details. □

By a reasoning similar to the proof of Theorem 2.4 we also can prove the following assertions.

Theorem 2.6. *Let* $u(x,y), a(x,y), b(x,y), k(x,y)$ *be nonnegative continuous functions in* R_+^2, *and let* $a(x,y)$ *be nondecreasing in* x *and nonincreasing in* y. *Suppose that*

$$u(x,y) \leq a(x,y) + \int_0^x b(s,y)u(s,y)\,ds + \int_0^x \int_y^\infty k(s,t)u^p(s,t)\,dtds$$

for $x \geq 0, y \geq 0$, *where* $p \geq 0, p \neq 1$, *is a constants, and*

$$\int_0^x \int_y^\infty k(s,t)\,dtds < \infty$$

for $x \geq 0, y \geq 0$. *Then*

$$u(x,y) \leq \exp\left(\int_0^x b(\tau,y)\,d\tau\right) \times \left[a^q(x,y) + q\int_0^x \int_y^\infty k(s,t)\exp\left(\int_0^s b(\tau,y)\,d\tau\right) dtds\right]^{1/q}$$

for $x \in [0,X), y \in [0,Y)$, *where* $q = 1-p$, X *and* Y *are chosen so that the expression between [...] is positive in the subintervals* $[0,X)$ *and* $[0,Y)$.

3. Further Inequalities

In this section we consider further nonlinear integral inequalities for functions of two independent variables.

Theorem 3.1. *Let $u(x,y), a(x,y), b(x,y), k(x,y)$ be nonnegative continuous functions for $x \geq x_0, y \geq y_0$, and let $a(x,y)$ be nondecreasing in each of the variables for $x \geq x_0, y \geq y_0$. Suppose that*

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x} b(s,y)u^p(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u^p(s,t)\,dtds \tag{3.1}$$

for $x \geq x_0, y \geq y_0$, where $p > 1$ is a constants and $\int_{x_0}^{x} b(s,y)u^p(s,y)\,ds$ be nondecreasing in y. Then

$$u(x,y) \leq \left[a^{1-p}(x,y) + (1-p)\left(\int_{x_0}^{x} b(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dtds\right)\right]^{(p-1)} \tag{3.2}$$

for $x \geq x_0, y \geq y_0$, and $(x,y) \in D$, where $D = \sup\{(x,y)|(1-p)(\int_{x_0}^{x} b(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dt\,ds) < a^{1-p}(x,y)\}$.

Proof. Define a function $v(x,y)$ by

$$v(x,y) = \int_{x_0}^{x} b(s,y)u^p(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u^p(s,t)\,dtds.$$

Then $v(x_0,y) = 0$, we have

$$\begin{aligned}
\frac{\partial v}{\partial x}(x,y) &\leq b(x,y)u^p(x,y) + \int_{y_0}^{y} k(x,t)u^p(x,t)\,dt \\
&\leq \left(b(x,y) + \int_{y_0}^{y} k(x,t)\,dt\right)[a(x,y) + v(x,y)]^p \\
&\leq \left(b(x,y) + \int_{y_0}^{y} k(x,t)\,dt\right)[a(x,y) + v(x,y)]^{(p-1)}[a(x,y) + v(x,y)] \qquad (3.3)
\end{aligned}$$

since $u(x,y) \leq a(x,y) + v(x,y)$. The inequality (3.3) imply that

$$\frac{\partial v}{\partial x}(x,y) \leq R(x,y)[a(x,y) + v(x,y)], \tag{3.4}$$

where $R(x,y) = (b(x,y) + \int_{y_0}^{y} k(x,t)\,dt)[a(x,y) + v(x,y)]^{(p-1)}$. Inequality (3.4) implies

$$\left[\frac{\partial v}{\partial s}(s,y) - R(s,y)v(s,y)\right]\exp\left(\int_{s}^{x} R(\tau,y)\,d\tau\right) \leq R(s,y)a(s,y)\exp\left(\int_{s}^{x} R(\tau,y)\,d\tau\right)$$

for $s \geq x_0$, or

$$\frac{\partial}{\partial s}\left[v(s,y)\exp\left(\int_s^x R(\tau,y)\,d\tau\right)\right] \leq R(s,y)a(s,y)\exp\left(\int_s^x R(\tau,y)\,d\tau\right).$$

Integration over s from x_0 to x gives

$$v(x,y) \leq \int_{x_0}^x R(s,y)a(s,y)\exp\left(\int_s^x R(\tau,y)\,d\tau\right)ds,$$

which implies

$$v(x,y) \leq a(x,y)\int_{x_0}^x R(s,y)\exp\left(\int_s^x R(\tau,y)\,d\tau\right)ds, \tag{3.5}$$

since $v(x_0,y)=0$. From (3.5), we get

$$v(x,y)+a(x,y) \leq a(x,y)\exp\left(\int_{x_0}^x R(\tau,y)\,d\tau\right). \tag{3.6}$$

From (3.6) we successively obtain

$$\begin{aligned}
&[v(x,y)+a(x,y)]^{(p-1)} \leq a^{(p-1)}(x,y)\exp\left((p-1)\int_{x_0}^x R(\tau,y)\,d\tau\right),\\
&R(x,y) \leq \left[b(x,y)+\int_{y_0}^y k(x,t)\,dt\right]a^{(p-1)}(x,y)\exp\left((p-1)\int_{x_0}^x R(\tau,y)\,d\tau\right),\\
&Z(x,y) = (p-1)R(x,y)\\
&\qquad \leq (p-1)\left[b(x,y)+\int_{y_0}^y k(x,t)\,dt\right]a^{(p-1)}(x,y)\exp\left(\int_{x_0}^x Z(\tau,y)\,d\tau\right).
\end{aligned}$$

Consequently

$$Z(x,y)\exp\left(-\int_{x_0}^x Z(\tau,y)\,d\tau\right) \leq (p-1)\left[b(x,y)+\int_{y_0}^y k(x,t)\,dt\right]a^{(p-1)}(x,y),$$

or

$$\frac{\partial}{\partial s}\left[-\exp\left(-\int_{x_0}^s Z(\tau,y)\,d\tau\right)\right] \leq (p-1)\left[b(s,y)+\int_{y_0}^y k(s,t)\,dt\right]a^{(p-1)}(s,y).$$

Integrating this from x_0 to x yields

$$1-\exp\Big(-\int_{x_0}^{x} Z(\tau,y)\,d\tau\Big) \le \int_{x_0}^{x}(p-1)\Big[b(s,y)+\int_{y_0}^{y}k(s,t)\,dt\Big]a^{(p-1)}(s,y)\,ds,$$

from which we conclude that

(3.7)
$$\exp\Big(\int_{x_0}^{x} R(\tau,y)\,d\tau\Big) \le \Big[1-(p-1)a^{(p-1)}(x,y)\int_{x_0}^{x}\Big(b(s,y)+\int_{y_0}^{y}k(s,t)\,dt\Big)\,ds\Big]^{(p-1)}$$

for $x \ge x_0, y \ge y_0$, and $(x,y) \in D$, where $D = \sup\{(x,y)|(1-p)(\int_{x_0}^{x} b(s,y)\,ds + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\,dt) < a^{1-p}(x,y)\}$. The desired inequality in (3.2) follows by using (3.6),(3.7) and the fact that $u(x,y) \le a(x,y)+v(x,y)$. $\square$

By a reasoning similar to the proof of Theorem 3.1 we also can prove the following assertions.

Theorem 3.2. *Let $u(x,y), a(x,y), b(x,y), k(x,y)$ be nonnegative continuous functions in R_+^2, and let $a(x,y)$ be nonincreasing in each of the variables in $x \ge 0, y \ge 0$. Suppose that*

$$u(x,y) \le a(x,y) + \int_{x}^{\infty} b(s,y)u^p(s,y)\,ds + \int_{x}^{\infty}\int_{y}^{\infty} k(s,t)u^p(s,t)\,dtds$$

for $x \ge 0, y \ge 0$, where $p > 1$ is a constants,

$$\int_{x}^{\infty} b(s,y)\,ds < \infty, \quad \int_{x}^{\infty}\int_{y}^{\infty} k(s,t)\,dtds < \infty,$$

and $\int_{x}^{\infty} b(s,y)u^p(s,y)\,ds$ be nonincreasing in y. Then

$$u(x,y) \le \Big[a^{1-p}(x,y) + (1-p)\Big(\int_{x}^{\infty} b(s,y)\,ds + \int_{x}^{\infty}\int_{y}^{\infty} k(s,t)\,dtds\Big)\Big]^{(p-1)}$$

for $x \ge 0, y \ge 0$, and $(x,y) \in D$ where $D = \sup\{(x,y)|(1-p)(\int_{x}^{\infty} b(s,y)\,ds + \int_{x}^{\infty}\int_{y}^{\infty} k(s,t)\,dt\,ds) < a^{1-p}(x,y)\}$.

Theorem 3.3. *Let $u(x,y), a(x,y), b(x,y), k(x,y)$ be nonnegative continuous functions in R^2_+, and let $a(x,y)$ be nondecreasing in $x, x \geq 0$, and nonincreasing in $y, y \geq 0$. Suppose that*

$$u(x,y) \leq a(x,y) + \int_0^x b(s,y)u^p(s,y)\,ds + \int_0^x \int_y^\infty k(s,t)u^p(s,t)\,dtds$$

for $x \geq 0, y \geq 0$, where $p > 1$ is a constants,

$$\int_0^x \int_y^\infty k(s,t)\,dtds < \infty,$$

and $\int_0^x b(s,y)u^p(s,y)\,ds$ be nonincreasing in y. Then

$$u(x,y) \leq \left[a^{1-p}(x,y) + (1-p)\left(\int_0^x b(s,y)\,ds + \int_0^x \int_y^\infty k(s,t)\,dtds\right)\right]^{(p-1)}$$

for $x \geq 0, y \geq 0$, and $(x,y) \in D$ where $D = \sup\{(x,y)|(1-p)(\int_0^x b(s,y)\,ds + \int_0^x \int_y^\infty k(s,t)\,dt\,ds) < a^{1-p}(x,y)\}$.

4. Applications

In this section we present some immediate applications of Theorem 2.5 to study certain properties of solutions of the following terminal value problem for the hyperbolic partial differential equation

(4.1) $$u_{xy}(x,y) = h(x,y,u(x,y)) + r(x,y),$$

(4.2) $$u(x,\infty) = \sigma_\infty(x), u(\infty,y) = \tau_\infty(y), u(\infty,\infty) = k,$$

where $h : R^2_+ \times R \to R, r : R^2_+ \to R, \sigma_\infty, \tau_\infty(y) : R_+ \to R$ are continuous functions and k is a real constant.

The following example deals with the estimate on the solution of the partial differential equation (4.1) with the conditions (4.2).

Example 1. Suppose that the function h in (4.1) satisfies the condition

(4.3) $$\mid h(x,y,u) \mid \leq k(x,y) \mid u \mid^p,$$

and

(4.4) $$\left|\sigma_\infty(x) + \tau_\infty(y) - k + \int_x^\infty \int_y^\infty r(s,t)\,dtds\right| \leq a(x,y) + \int_x^\infty b(s,y)u(s,y)\,ds,$$

where $a(x,y), b(x,y), k(x,y)$ are as defined in Theorem 2.5. If $u(x,y)$ be a solution of (4.1) with the conditions (4.2), then it can be written as (see [1, p. 80])

$$u(x,y) = \sigma_\infty(x) + \tau_\infty(y) - k + \int_x^\infty \int_y^\infty \big(h(s,t,u(s,t)) + r(s,t)\big)\, dtds \tag{4.5}$$

for $x, y \in R$. From (4.3), (4.4), (4.5) we get

$$|u(x,y)| \le a(x,y) + \int_x^\infty b(s,y)|u|\, ds + \int_x^\infty \int_y^\infty k(s,t)|u|^p\, dtds. \tag{4.6}$$

Now, a suitable application of Theorem 2.5 to (4.6) yields the required estimate following

$$\begin{aligned} |u(x,y)| \le & \exp\left(\int_x^\infty b(\tau,y)\, d\tau\right) \\ & \times \left[a^q(x,y) + q\int_x^\infty \int_y^\infty k(s,t) \exp\left(\int_s^\infty b(\tau,y)\, d\tau\right) dtds\right]^{1/q} \end{aligned} \tag{4.7}$$

for $x \in [0, X), y \in [0, Y)$, where $q = 1-p$, X and Y are chosen so that the expression between [...] is positive in the subintervals $[0, X)$ and $[0, Y)$. The right-hand side of (4.7) gives us the bound on the solution $u(x,y)$ of (4.1)-(4.2) in terms of the known functions. Thus, if the right-hand side of (4.7) is bounded, then we assert that the solution of (4.1)-(4.2) is bounded for $x \in [0, X), y \in [0, Y)$.

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