BOUNDING MATHIEU TYPE SERIES

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Abstract. An integral representation for a generalised Mathieu series is used to obtain bounds. The bounds are obtained using results pertaining to the Čebyšev functional.

1. Introduction

The series, known in the literature as the Mathieu series,

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^2},\quad r>0, \tag{1.1}$$

has been extensively studied in the past since its introduction by Mathieu [17] in 1890, where it arose in connection with work on elasticity of solid bodies. The reader is directed to the references for further illustration.

One of the main questions addressed in relation (1.1) is to obtain sharp bounds. Alzer, Brenner and Ruehr [2] showed that the best constants a and b in

$$\frac{1}{x^2+a}<S(x)<\frac{1}{x^2+b},\qquad x\neq 0 \tag{1.2}$$

are $a=\frac{1}{2\zeta(3)}$ and $b=\frac{1}{6}$ where $\zeta(\cdot)$ denotes the Riemann zeta function defined by

$$\zeta(p)=\sum_{n=1}^{\infty}\frac{1}{n^p}. \tag{1.3}$$

An integral representation for $S(r)$ as given in (1.1) was presented in [10] and [11] as

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{x}{e^x-1}\sin(rx)\,dx. \tag{1.4}$$

Guo [14] utilised (1.4) to obtain bounds on $S(r)$. Alternate bounds to (1.1) were obtained by Qi and coworkers in [19, 20, 21].

Guo in [14] poses the interesting problem as to whether there is an integral representation of the generalised Mathieu series

$$S_\mu(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^{1+\mu}},\qquad r>0,\ \mu>0. \tag{1.5}$$

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Recently in [22] an integral representation was obtained for $S_m(r)$, where $m \in \mathbb{N}$, namely

$$(1.6)\quad S_m(r) = \frac{2}{(2r)^m m!}\int_0^\infty \frac{t^m}{e^t-1}\cos\left(\frac{m\pi}{2}-rt\right)dt - 2\sum_{k=1}^{m}\left[\frac{(k-1)(2r)^{k-2m-1}}{k!(m-k+1)}\binom{-(m+1)}{m-k} \times \int_0^\infty \frac{t^k\cos\left[\frac{\pi}{2}(2m-k+1)-rt\right]}{e^t-1}dt\right].$$

The challenge of Guo [14] to obtain an integral representation for $S_\mu(r)$ as defined in (1.5) has been successfully answered by Cerone and Lenard [6] in which the following two theorems were proved.

Theorem 1. *The generalised Mathieu series $S_\mu(r)$ defined by (1.5) may be represented in the integral form*

$$(1.7)\quad S_\mu(r) = C_\mu(r)\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1}J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad \mu>0,$$

where

$$(1.8)\quad C_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)}$$

and $J_\nu(z)$ is the ν^{th} order Bessel function of the first kind.

Theorem 2. *For m a positive integer we have*

$$(1.9)\quad S_m(r) = \frac{1}{2^{m-1}}\cdot\frac{1}{r^{2m-1}}\cdot\frac{1}{m}\sum_{k=0}^{m-1}\frac{(-1)^{\lfloor\frac{3k}{2}\rfloor}}{k!}r^k\left[\delta_{k\ even}A_k(r)+\delta_{k\ odd}B_k(r)\right],$$

where

$$(1.10)\quad A_k(r) = \int_0^\infty \frac{x^{k+1}}{e^x-1}\sin(rx)\,dx, \quad B_k(r) = \int_0^\infty \frac{x^{k+1}}{e^x-1}\cos(rx)\,dx,$$

with $\delta_{condition}=1$ if condition holds and zero otherwise and $\lfloor x\rfloor$ is the smallest integer part of x.

The emphasis now becomes to obtain bounds for the generalised Mathieu series $S_\mu(r)$. The first approach is to utilise sharp bounds for the Bessel function $|J_\nu(z)|$. To this end, in a recent article Landau [15] obtained the best possible uniform bounds for Bessel functions using monotonicity arguments. Of particular interest to us here is that he showed

$$(1.11)\quad |J_\nu(z)| < \frac{b_L}{\nu^{\frac{1}{3}}}$$

uniformly in the argument $z>0$ and is best possible in the exponent $\frac{1}{3}$ and constant

$$(1.12)\quad b_L = 2^{\frac{1}{3}}\sup_x Ai(z) = 0.674885\cdots,$$

where $Ai(z)$ is the Airy function satisfying

$$w'' - zw = 0.$$

Landau also showed that for $z > 0$

$$|J_\nu(z)| \le \frac{c_L}{z^{\frac{1}{3}}} \tag{1.13}$$

uniformly in the order $\nu > 0$ and the exponent $\frac{1}{3}$ is best possible with

$$\begin{aligned} c_L &= \sup_z z^{\frac{1}{3}} J_0(z) \\ &= 0.78574687\ldots. \end{aligned} \tag{1.14}$$

The following theorem, based on the Landau bounds (1.11) – (1.14), was obtained in [6].

Theorem 3. *The generalised Mathieu series $S_\mu(r)$ satisfies the bounds for $\mu > \frac{1}{2}$ and $r > 0$*

$$S_\mu(r) \le b_L \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}} \cdot \frac{1}{\left(\mu - \frac{1}{2}\right)^{\frac{1}{3}}} \cdot \frac{\Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma(\mu+1)} \zeta\left(\mu + \frac{3}{2}\right), \tag{1.15}$$

and

$$S_\mu(r) \le c_L \cdot \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}} r^{\mu-\frac{1}{6}}} \cdot \Gamma\left(\mu + \frac{7}{6}\right) \zeta\left(\mu + \frac{7}{6}\right), \tag{1.16}$$

where b_L and c_L are given by (1.12) and (1.14) respectively.

The following corollary was also obtained in [6] for $S_1(r)$.

Corollary 1. *The Mathieu series $S(r)$ satisfies the following bounds*

$$S(r) \le \frac{3\pi}{2^{\frac{11}{12}}} b_L \cdot \zeta\left(\frac{5}{2}\right) \tag{1.17}$$

and

$$S(r) \le \frac{7c_L}{36} \cdot \sqrt{\frac{\pi}{2}} \cdot \Gamma\left(\frac{1}{6}\right) \zeta\left(\frac{13}{6}\right) \cdot r^{-\frac{5}{6}}, \tag{1.18}$$

where b_L and c_L are given by (1.12) and (1.14) respectively.

In the current paper, further bounds are obtained for the generalised Mathieu series $S_\mu(r)$ utilising some recent results on bounding the integral of a product of functions.

2. Some Results on Bounding the Čebyšev Functional

The weighted Čebyšev functional defined by

$$T(f, g; p) := \mathcal{M}(fg; p) - \mathcal{M}(f; p)\, \mathcal{M}(g; p), \tag{2.1}$$

where

$$\mathcal{M}(f; p) := \frac{\int_a^b p(x)\, h(x)\, dx}{\int_a^b p(x)\, dx}, \tag{2.2}$$

the weighted integral mean, has been extensively investigated in the literature with the view of determining its bounds.

There has been much activity in procuring bounds for $T(f,g;p)$ and the interested reader is referred to [4, 5]. The functional $T(f,g;p)$ is known to satisfy a number of identities. Included amongst these are identities of Sönin type, namely

$$P\cdot T(f,g;p)=\int_a^b p(t)\left[f(t)-K\right]\left[g(t)-\mathcal{M}(g;p)\right]dt,\ \text{for } K \text{ a constant.} \tag{2.3}$$

The constant $K\in\mathbb{R}$ but in the literature some of the more popular values have been taken as

$$0,\ \frac{\Delta+\delta}{2},\ f\left(\frac{a+b}{2}\right) \text{ and } \mathcal{M}(f;p),$$

where $-\infty<\delta\le f(t)\le\Delta<\infty$ for $t\in[a,b]$.

An identity attributed to Körkine viz

$$P^2\cdot T(f,g;p)=\frac{1}{2}\int_a^b\int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy \tag{2.4}$$

may also easily be shown to hold.

Remark 1. *For $-\infty<\delta\le f(t)\le\Delta<\infty$ for $t\in[a,b]$ Cerone and Dragomir* [5] *showed that*

$$\begin{aligned} P\cdot|T(f,g;p)| &\le \frac{1}{2}(\Delta-\delta)\int_a^b p(t)\left|g(t)-\mathcal{M}(g;p)\right|dt \\ &\le \frac{1}{2}(\Delta-\delta)\left(\int_a^b p(t)\left|g(t)-\mathcal{M}(g;p)\right|^{\alpha}dt\right)^{\frac{1}{\alpha}},\ 1\le\alpha<\infty \\ &\le \frac{1}{2}(\Delta-\delta)\,ess\sup_{t\in[a,b]}\left|g(t)-\mathcal{M}(g;p)\right|. \end{aligned} \tag{2.5}$$

Specifically, if $-\infty<\phi\le g(t)\le\Phi<\infty$ for $t\in[a,b]$, then

$$\begin{aligned} |T(f,g;p)| &\le \frac{1}{2}(\Delta-\delta)\int_a^b p(t)\left|g(t)-\mathcal{M}(g;p)\right|dt \\ &\le \frac{1}{2}(\Delta-\delta)\left[\frac{1}{P}\int_a^b p(t)g^2(t)\,dt-\mathcal{M}^2(g;p)\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Phi-\phi). \end{aligned} \tag{2.6}$$

The results in (2.5) were obtained from the Sönin type identity (2.3) on taking $K=\frac{\Delta+\delta}{2}$.

It is instructive to show from (2.3) that the best K, in the sense of providing the sharpest bound for the Euclidean or $2-$norm results when $K=\mathcal{M}(f;p)$.

Lemma 1. *The sharpest bound for the Čebyšev functional involving the Euclidean norm is given by*

$$
\begin{aligned}
(2.7)\qquad & P\cdot\left|T\left(f,g;p\right)\right| \\
& \leq \inf_{K}\left[\int_a^b p\left(t\right)\left(f\left(t\right)-K\right)^2 dt\right]^{\frac{1}{2}}\left[\int_a^b p\left(t\right)\left(g\left(t\right)-\mathcal{M}\left(g;p\right)\right)^2 dt\right]^{\frac{1}{2}} \\
& = \left[\int_a^b p\left(t\right)f^2\left(t\right)dt-\mathcal{M}^2\left(f;p\right)\right]^{\frac{1}{2}}\left[\frac{1}{P}\int_a^b p\left(t\right)g^2\left(t\right)dt-\mathcal{M}^2\left(g;p\right)\right]^{\frac{1}{2}}.
\end{aligned}
$$

Proof. From (2.3) we have, on using the Cauchy-Buniakowsky-Schwartz inequality that

$$
P\cdot\left|T\left(f,g;p\right)\right|\leq\left(\int_a^b p\left(t\right)\left(f\left(t\right)-K\right)^2 dt\right)^{\frac{1}{2}}\left(\int_a^b p\left(t\right)\left(g\left(t\right)-\mathcal{M}\left(g;p\right)\right)^2 dt\right)^{\frac{1}{2}}.
$$

Now, the sharpest bound is obtained by taking the infimum over $K\in\mathbb{R}$. That is

$$
\begin{aligned}
\inf_{K\in\mathbb{R}}\left(\int_a^b p\left(t\right)\left(f\left(t\right)-K\right)^2 dt\right)^{\frac{1}{2}} & = \inf_{K\in\mathbb{R}}\left(\int_a^b p\left(t\right)\left(f^2\left(t\right)-2Kf\left(t\right)+K^2\right)dt\right)^{\frac{1}{2}} \\
& = \inf_{K\in\mathbb{R}}\left[\int_a^b p\left(t\right)f^2\left(t\right)dt+P\cdot K\left(K-2\mathcal{M}\left(f;p\right)\right)\right]^{\frac{1}{2}} \\
& = \left(\int_a^b p\left(t\right)f^2\left(t\right)dt-P\cdot\mathcal{M}^2\left(f;p\right)\right)^{\frac{1}{2}},
\end{aligned}
$$

where $K=\mathcal{M}\left(f;p\right).$ ∎

In the next section Lemma 1 is used to obtain bounds for the generalised Mathieu series $S_\mu\left(r\right).$

We note that the first inequality in (2.5) results from

$$
\begin{aligned}
(2.8)\qquad \left|P\cdot T\left(f,g;p\right)\right| & \leq \inf_{K}\left\|f\left(\cdot\right)-K\right\|_\infty\int_a^b p\left(t\right)\left|g\left(t\right)-\mathcal{M}\left(g;p\right)\right|dt \\
& \leq \left\|f\left(\cdot\right)-K\right\|_\infty\int_a^b p\left(t\right)\left|g\left(t\right)-\mathcal{M}\left(g;p\right)\right|dt,
\end{aligned}
$$

which are tighter than those in Lemma 1.

However, (2.8) relies on knowing where the shifted functions are positive and where they are negative. This is not always an easy task.

The first result in (2.5) arises from (2.8) with $K=\frac{\Delta+\delta}{2}$ so that

$$
\left\|f\left(\cdot\right)-\frac{\Delta+\delta}{2}\right\|_\infty=\sup_{t\in[a,b]}\left|f\left(t\right)-\frac{\Delta+\delta}{2}\right|=\frac{\Delta-\delta}{2},
$$

where $-\infty<\delta\leq f\left(t\right)\leq\Delta<\infty$ for $t\in\left[a,b\right].$

3. Bounds for $S_\mu(r)$ via the Čebyšev Functional

Bounds on the Čebyšev functional (2.1) may be looked upon as estimating the distance of the weighted mean of the product of two functions from the product of the weighted mean of the two functions. This proves to be quite useful since the individual means are invariably easier to evaluate.

Here we investigate the bounding of $S_\mu(r)$ as defined by (1.5) through the identities (1.7) – (1.8). We notice that bounding $S_\mu(r)$ is accomplished via $\chi_\mu(r)$ where

$$\chi_\mu(r) := \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1} J_{\mu-\frac{1}{2}}(rx)\,dx; \qquad \mu, r > 0, \tag{3.1}$$

since from (1.7)

$$S_\mu(r) = C_\mu(r)\,\chi_\mu(r), \tag{3.2}$$

where $C_\mu(r)$ is positive as defined in (1.8).

The following lemma examines the behaviour of $\chi_\mu(r)$.

Lemma 2.

$$\left|\chi_\mu(r) - \frac{1}{2}\cdot\frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}}\cdot\frac{\Gamma(\mu)}{\left(r^2+\frac{1}{4}\right)^\mu}\cdot\frac{\pi^2}{6}\right| \tag{3.3}$$
$$\leq \kappa\left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right) r^{2\mu-1}}{\pi^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi - 2K_*^2\right]^{\frac{1}{2}},$$

where

$$K_* = \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{2\sqrt{\pi}\left(r^2+\frac{1}{4}\right)^\mu} \quad \textit{is defined in (3.13),}$$

and

$$\kappa = \left[\pi^2\left(1-\frac{\pi^2}{72}\right) - 7\zeta(3)\right]^{\frac{1}{2}} = 0.319846901\ldots. \tag{3.4}$$

Proof. Firstly, we notice that $\chi_\mu(r)$ from (3.1) may be written in the form

$$\chi_\mu(r) = \int_0^\infty e^{-\frac{x}{2}}\cdot\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx. \tag{3.5}$$

Let

$$p(x) = e^{-\frac{x}{2}}, \quad f(x) = \frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}, \quad g(x) = x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) \tag{3.6}$$

then from (2.2)

$$P = \int_0^\infty p(x)\,dx = \int_0^\infty e^{-\frac{x}{2}}\,dx = 2, \tag{3.7}$$

$$P\cdot\mathcal{M}(f;p) = \int_0^\infty e^{-\frac{x}{2}}\cdot\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}dx = \int_0^\infty \frac{x}{e^x-1}dx = \zeta(2) = \frac{\pi^2}{6} \tag{3.8}$$

and

$$P \cdot \mathcal{M}(g;p) = \int_0^\infty e^{-\frac{x}{2}} \cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx = \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}\left(\left(\frac{1}{2}\right)^2 + r^2\right)^{\mu}}, \tag{3.9}$$

where we have used the fact that [13]

$$\Gamma(p+1)\zeta(p+1) = \int_0^\infty \frac{x^p}{e^x - 1}dx \text{ to procure (3.8)}, \tag{3.10}$$

and Watson [24, p. 386]

$$\int_0^\infty e^{-\alpha x} \cdot x^\nu J_\nu(\beta x)\,dx = \frac{(2\beta)^\nu}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\left(\alpha^2 + \beta^2\right)^{\nu+\frac{1}{2}}}, \quad \mathrm{Re}(\nu) > \frac{1}{2}, \quad \mathrm{Re}(\alpha) > |\mathrm{Im}(\beta)|,$$

with $\alpha = \frac{1}{2}$, $\nu = \mu - \frac{1}{2}$, $\beta = r$ to obtain (3.9).

Now, from (2.1) – (2.3) we have on using (3.6) – (3.9)

$$\begin{aligned} \chi_\mu(r) - \frac{1}{2} \cdot \frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma(\mu)}{\left(r^2 + \frac{1}{4}\right)^\mu} \cdot \frac{\pi^2}{6} & \\ = \int_0^\infty e^{-\frac{x}{2}} \left(x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) - K\right) \left(\frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} - \frac{\pi^2}{12}\right) dx. & \end{aligned} \tag{3.11}$$

Now, by using the Cauchy-Buniakowsky-Schwartz inequality, we have from (3.11)

$$\begin{aligned} \left|\chi_\mu(r) - \frac{1}{2} \cdot \frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma(\mu)}{\left(r^2 + \frac{1}{4}\right)^\mu} \cdot \frac{\pi^2}{6}\right| & \\ \le \left(\int_0^\infty e^{-\frac{x}{2}} \left(x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) - K\right)^2 dx\right)^{\frac{1}{2}} & \\ \times \left(\int_0^\infty e^{-\frac{x}{2}} \left(\frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} - \frac{\pi^2}{12}\right)^2 dx\right)^{\frac{1}{2}}. & \end{aligned} \tag{3.12}$$

As mentioned in Section 2, the appropriate choice of K is the weighted integral mean as given from (3.9), namely

$$K = K_* = \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{2\sqrt{\pi}\left(r^2 + \frac{1}{4}\right)^\mu}. \tag{3.13}$$

It may be easily shown by expansion that

$$\int_a^b p(t)\left[h(t) - \mathcal{M}(h;p)\right]^2 dt = \int_a^b p(t)\,h^2(t)\,dt - P \cdot \mathcal{M}^2(h;p). \tag{3.14}$$

The result (3.14) was a by product of the proof of Lemma 1.

This result will be utilised to evaluate the two expressions on the right hand side of (3.12).

Thus from (3.12) we have

$$\int_0^\infty e^{-\frac{x}{2}} \left(\frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} - \frac{\pi^2}{12}\right)^2 dx = \int_0^\infty e^{-\frac{x}{2}} \left(\frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}\right)^2 dx - 2\left(\frac{\pi^2}{12}\right)^2. \tag{3.15}$$

Now, allowing for the permissable interchange of integration and summation, we have

$$\begin{aligned}(3.16)\qquad \int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)^2 dx &= \int_0^\infty e^{-\frac{3x}{2}}\left(\frac{x}{1-e^{-x}}\right)^2 dx \\ &= \int_0^\infty e^{-\frac{3x}{2}} x^2 \left(\sum_{n=1}^\infty n e^{-nx}\right) dx \\ &= \sum_{n=1}^\infty n \int_0^\infty e^{-\left(\frac{2n+1}{2}\right)x} x^2 dx \\ &= \sum_{n=1}^\infty \frac{n\Gamma(3)}{\left(\frac{2n+1}{2}\right)^3} = \sum_{n=1}^\infty \frac{2n}{\left(n+\frac{1}{2}\right)^3} \\ &= 2\sum_{n=1}^\infty \frac{1}{\left(n+\frac{1}{2}\right)^2} - \sum_{n=1}^\infty \frac{1}{\left(n+\frac{1}{2}\right)^3} \\ &= \pi^2 - 7\cdot\zeta(3).\end{aligned}$$

In (3.16) we have used the fact that

$$\int_0^\infty e^{-\alpha x} x^p dx = \frac{\Gamma(p+1)}{\alpha^{p+1}}.$$

Hence, from (3.15) and (3.16) we have

$$(3.17)\qquad \left[\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}-\frac{\pi^2}{12}\right)^2 dx\right]^{\frac{1}{2}} = \left[\pi^2\left(1-\frac{\pi^2}{72}\right)-7\zeta(3)\right]^{\frac{1}{2}}.$$

Now, for the first expression on the right hand side of (3.12), we have, on using (3.13) and(3.14)

$$(3.18)\quad \int_0^\infty e^{-\frac{x}{2}}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)-K_*\right)^2 dx = \int_0^\infty e^{-\frac{x}{2}} x^{2\mu-1} J^2_{\mu-\frac{1}{2}}(rx)\,dx - 2K_*^2.$$

A result in Watson [24, p. 290] states that

$$\begin{aligned}(3.19)\quad &\int_0^\infty e^{-2at} J_\alpha(\gamma t) J_\beta(\gamma t)\, t^{\alpha+\beta} dt \\ &\qquad = \frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)}{\pi^{\frac{3}{2}}}\gamma^{\alpha+\beta}\int_0^{\frac{\pi}{2}} \frac{\cos^{\alpha+\beta}\phi\cos(\alpha-\beta)\phi}{\left(a^2+\gamma^2\cos^2\phi\right)^{\alpha+\beta+\frac{1}{2}}} d\phi\end{aligned}$$

and so taking $a=\frac{1}{4}$, $\alpha=\beta=\mu-\frac{1}{2}$ and $\gamma=r$ in (3.19) gives

$$\begin{aligned}(3.20)\quad &\int_0^\infty e^{-\frac{x}{2}} x^{2\mu-1} J^2_{\mu-\frac{1}{2}}(rx)\,dx \\ &\qquad = \frac{\Gamma\left(2\mu-\frac{1}{2}\right) r^{2\mu-1}}{\pi^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left(\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right)^{2\mu-\frac{1}{2}}} d\phi\end{aligned}$$

That is,

$$\text{(3.21)}\qquad \left[\int_0^\infty e^{-\frac{x}{2}}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)-K_*\right)^2 dx\right]^{\frac{1}{2}}$$

$$=\left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{\pi^{\frac{3}{2}}}r^{2\mu-1}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi-2K_*^2\right]^{\frac{1}{2}}.$$

Placing (3.21) and (3.17) into (3.12) produces the stated result (3.3).

For the coarser bound in (3.3) we notice that

$$\text{(3.22)}\qquad \int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi<\frac{1}{r^{4\mu-1}}\int_0^\infty\frac{d\phi}{\cos^{2\mu}\phi}$$

$$=\frac{1}{r^{4\mu-1}}\cdot\frac{\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(\mu+1)}$$

and so on substitution into the first result in (3.3) produces the second upon some simplification. ∎

Theorem 4. *For $\mu>0$ and $r>0$ the generalised Mathieu series $S_\mu(r)$ satisfies the following relationship, namely,*

$$\text{(3.23)}\qquad \left|S_\mu(r)-\frac{\pi^2}{12\mu\left(r^2+\frac{1}{4}\right)^\mu}\right|$$

$$\le\kappa\left[\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\Gamma^2(\mu+1)}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi-\frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}}\right]^{\frac{1}{2}}$$

$$\le\kappa\left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)}{2^{2\mu}\Gamma^3(\mu+1)}\cdot\frac{1}{r^{4\mu-1}}-\frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}}\right]^{\frac{1}{2}},$$

where κ is as given by (3.4).

Proof. From (3.2) we have, since $C_\mu(r)$, as defined by (1.8), is a positive so that using Lemma 2 readily produces the above results (3.23) upon simplification. ∎

Corollary 2. *The following bounds are valid for $S(r)$ the Mathieu series. That is,*

$$\text{(3.24)}\qquad \left|\sum_{n=1}^\infty\frac{2n}{(n^2+r^2)^2}-\frac{\pi^2}{12\left(r^2+\frac{1}{4}\right)}\right|\le2\sqrt{2}\cdot\kappa\left\{\frac{2}{1+(4r)^2}-\frac{1}{\left[1+(2r)^2\right]^2}\right\}^{\frac{1}{2}},$$

where κ is as given by (3.4).

Proof. Let $\mu=1$ in (3.23) and using (1.1) and (1.5) gives the above result (3.24), on noting that

$$2^6\int_0^{\frac{\pi}{2}}\frac{\cos\phi}{\left[1+(4r\cos\phi)^2\right]^{\frac{3}{2}}}d\phi=\frac{64}{1+(4r)^2}$$

and after some simplification. ∎

Remark 2. *The result of Theorem 4 holds for any $\mu > 0$ and $r > 0$ whereas those obtained in* [6] *were valid for $\mu > \frac{1}{2}$.*

Remark 3. *From (3.17) we may infer*

$$\zeta(3) < \frac{\pi^2}{7}\left(1 - \frac{\pi^2}{72}\right) = 1.21667148\ldots. \tag{3.25}$$

Guo [14] *obtains $\zeta(3) < \frac{\pi^4}{72} = 1.353904\ldots$ and so (3.25) is an improvement.*

4. Some Properties of $S_\mu(r)$

The generalised Mathieu series $S_\mu(r)$ is a positive, decreasing function of both μ and r for $\mu > 0$, $r > 0$.

The following interesting result holds.

Proposition 1. *The generalised Mathieu series as defined in (1.5) satisfies the identity*

$$\int_0^\infty S_\mu(r)\,dr = \sqrt{\pi}\cdot\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\mu\Gamma(\mu)}\zeta(2\mu), \quad \mu > 0. \tag{4.1}$$

Proof. Integrating (1.7) with respect to r and noting (1.8) gives

$$\int_0^\infty S_\mu(r)\,dr = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+1)}\int_0^\infty\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-1}\cdot\frac{J_{\mu-\frac{1}{2}}(rx)}{r^{\mu-\frac{1}{2}}}dxdr. \tag{4.2}$$

Undertaking the permissable interchange of the order of integration and then making the substitution $w = xr$ produces from (4.2),

$$\int_0^\infty S_\mu(r)\,dr = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+1)}\int_0^\infty \frac{x^{2\mu-1}}{e^x-1}\left(\int_0^\infty \frac{J_{\mu-\frac{1}{2}}(w)}{w^{\mu-\frac{1}{2}}}dw\right)dx. \tag{4.3}$$

Now, using the fact that

$$\int_0^\infty x^{-\nu}J_\nu(x)\,dx = \frac{2^{-\nu}\Gamma(\pi)}{\Gamma\left(\nu+\frac{1}{2}\right)}$$

and (3.10) in (4.3)

$$\int_0^\infty S_\mu(r)\,dr = \frac{\pi}{2^{2\mu-1}}\cdot\frac{\Gamma(2\mu)}{\mu\Gamma^2(\mu)}\zeta(2\mu)$$

and on using the duplication formula for the gamma function

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

readily produces the stated result (4.1). ∎

Corollary 3. *For m a positive integer, then*

$$\int_0^\infty S_m(r)\,dr = \frac{(-1)^{m-1}2^{2m-1}\pi^{2m+\frac{1}{2}}}{m!\,(2m)!}\Gamma\left(m+\frac{1}{2}\right)B_{2m}, \tag{4.4}$$

where B_k are the Bernoulli numbers defined by

$$\frac{x}{e^x-1} = \sum_{k=0}^\infty \frac{x^k}{k!}B_k, \quad |x| < 2\pi.$$

Proof. Taking $\mu = m \in \mathbb{N}$ in (4.1) gives

$$\int_0^\infty S_m(r)\,dr = \frac{\sqrt{\pi}\Gamma\left(m+\frac{1}{2}\right)}{m!}\zeta(2m). \tag{4.5}$$

Now, a 1748 result of Euler states that for $m \in \mathbb{N}$

$$\zeta(2m) = (-1)^{m-1}\frac{2^{2m-1}\pi^{2m}}{(2m)!}B_{2m}. \tag{4.6}$$

Substitution of (4.6) in (4.5) readily produces the result (4.4). ∎

Remark 4. *If we take $\mu = 1$ in (4.1) (or alternatively, $m = 1$ in (4.4)), then we recapture the result of Guo* [14]*, namely*

$$\int_0^\infty S_1(r)\,dr = \frac{\pi^3}{12}. \tag{4.7}$$

Remark 5. *An alternative representation to (4.6) is given in a 1999 paper by Lin in Chinese (see* [16]*), namely,*

$$\zeta(2m) = A_m\pi^{2m}, \tag{4.8}$$

where A_m statisfies the recurrence relation

$$A_m = (-1)^{m-1}\cdot\frac{m}{(2m+1)!} + \sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(2j+1)!}A_{m-j} \tag{4.9}$$

and by convention the sum is neglected for $m = 1$ so that $A_1 = \frac{1}{3!}$.
Thus an equivalent result to (4.4) may be obtained as, from (4.5) and (4.8),

$$\int_0^\infty S_m(r)\,dr = \frac{\Gamma\left(m+\frac{1}{2}\right)}{m!}\pi^{2m+\frac{1}{2}}\cdot A_m$$

with A_m being given by (4.9).

Remark 6. *Guo* [14] *obtained the bound for $\zeta(3)$ stated in Remark 3 by using (4.7) and (1.2), where $S_1(r)$ is equivalent to $S(r)$.*

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