GENERALIZED RELATIVE INFORMATION AND INFORMATION INEQUALITIES

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ABSTRACT. In this paper, we have obtained bounds on Csiszár's *f-divergence* in terms of *relative information of type s* using Dragomir's [8] approach. The results obtained in particular lead us to bounds in terms of χ^2-*Divergence*, Kullback-Leibler's *relative information* and Hellinger's *discrimination.*

1. Introduction

Let

$$\Delta_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geqslant 2,$$

be the set of complete finite discrete probability distributions.

The Kullback Leibler's [11] *relative information* is given by

$$K(P||Q) = \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}), \tag{1.1}$$

for all $P, Q \in \Delta_n$.

In Δ_n, we have taken all $p_i > 0$. If we take $p_i \geqslant 0$, $\forall i = 1, 2, ..., n$, then in this case we have to suppose that $0 \ln 0 = 0 \ln(\frac{0}{0}) = 0$. From the *information theoretic* point of view we generally take all the logarithms with base 2, but here we have taken only natural logarithms.

We can observe that the measure (1.1) is not symmetric in P and Q. Its symmetric version famous as *J-divergence* (Jeffreys [10]; Kullback and Leiber [11]) is given by

$$J(P||Q) = K(P||Q) + K(Q||P) = \sum_{i=1}^{n} (p_i - q_i) \ln(\frac{p_i}{q_i}). \tag{1.2}$$

Let us consider the one parametric generalization of the measure (1.1), called *relative information of type s* given by

$$K_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1 \right], \quad s \neq 0, 1. \tag{1.3}$$

In this case we have the following limiting cases

$$\lim_{s \to 1} K_s(P||Q) = K(P||Q),$$

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and

$$\lim_{s\to 0} K_s(P||Q) = K(Q||P).$$

The expression (1.3) has been studied by Vajda [20]. Previous to it many authors studied its characterizations and applications (ref. Taneja [18] and on line book Taneja [19]).

We have some interesting particular cases of the measure (1.3).

(i) When $s = \frac{1}{2}$, we have

$$K_{1/2}(P||Q) = 4\,[1 - B(P||Q)] = 4\;h(P||Q)$$

where

$$B(P||Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}, \tag{1.4}$$

is the famous as Bhattacharya's [1] *distance*, and

$$h(P||Q) = \frac{1}{2}\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2, \tag{1.5}$$

is famous as Hellinger's [9] *discrimination*.

(ii) When $s = 2$, we have

$$K_2(P||Q) = \frac{1}{2}\chi^2(P||Q),$$

where

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1, \tag{1.6}$$

is the χ^2-*divergence* (Pearson [14]).

For simplicity, let us write the measures (1.3) in the unified way:

$$\Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0,1 \\ K(Q||P), & s = 0 \\ K(P||Q), & s = 1 \end{cases}, \tag{1.7}$$

2. Csiszár's $f-$Divergence and Information Bounds

Given a convex function $f : [0,\infty) \to \mathbb{R}$, the $f-$divergence measure introduced by Csiszár [4] is given by

$$C_f(p,q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{2.1}$$

where $p, q \in \mathbb{R}^n_+$.

The following two theorems can be seen in Csiszár and Körner [5].

Theorem 2.1. *(Joint convexity). If $f : [0,\infty) \to \mathbb{R}$ be convex, then $C_f(p,q)$ is jointly convex in p and q, where $p, q \in \mathbb{R}^n_+$.*

Theorem 2.2. *(Jensen's inequality). Let $f : [0, \infty) \to \mathbb{R}$ be a convex function. Then for any $p, q \in \mathbb{R}^n_+$, with $P_n = \sum_{i=1}^{n} p_i > 0, Q_n = \sum_{i=1}^{n} p_i > 0$, we have the inequality*

$$C_f(p, q) \geqslant Q_n f\left(\frac{P_n}{Q_n}\right).$$

The equality sign hold iff

$$\frac{p_1}{q_i} = \frac{p_2}{q_2} = ... = \frac{p_n}{q_n}.$$

In particular, for all $P, Q \in \Delta_n$, we have

$$C_f(P||Q) \geqslant f(1),$$

with equality iff $P = Q$.

In view of Theorems 2.1 and 2.2, we have the following result.

Result 2.3. *For all $P, Q \in \Delta_n$, we have*

(i) $\Phi_s(P||Q) \geqslant 0$ *for any $s \in \mathbb{R}$, with equality iff $P = Q$.*
(ii) $\Phi_s(P||Q)$ *is convex function of the pair of distributions $(P, Q) \in \Delta_n \times \Delta_n$ and for any $s \in \mathbb{R}$.*

Proof. Take

$$\phi_s(u) = \begin{cases} [s(s-1)]^{-1} \left[u^s - 1 - s(u-1)\right], & s \neq 0, 1 \\ u - 1 - \ln u, & s = 0 \\ 1 - u + u \ln u, & s = 1 \end{cases} \tag{2.2}$$

for all $u > 0$ in (2.1), we have

$$C_f(P||Q) = \Phi_s\left(P||Q\right) = \begin{cases} K_s(P||Q), & s \neq 0, 1 \\ K(Q||P), & s = 0 \\ K(P||Q), & s = 1 \end{cases}.$$

Moreover,

$$\phi_s'(u) = \begin{cases} (s-1)^{-1}\left(u^{s-1} - 1\right), & s \neq 0, 1 \\ 1 - u^{-1}, & s = 0 \\ \ln u, & s = 1 \end{cases}, \tag{2.3}$$

and

$$\phi_s''(u) = \begin{cases} u^{s-2}, & s \neq 0, 1 \\ u^{-2}, & s = 0 \\ u^{-1}, & s = 1 \end{cases} \tag{2.4}$$

Thus we have $\phi_s''(u) > 0$ for all $u > 0$, and hence, $\phi_s(u)$ is convex for all $u > 0$. Also, we have $\phi_s(1) = 0$. In view of Theorems 2.1 and 2.2 we have the proof of parts (i) and (ii) respectively. □

For some studies on the measure (2.2) refer to Liese and Vajda [13], Österreicher [15] and Cerone et al. [3].

The following theorem summarizes some of the results studies by Dragomir [6], [7]. For simplicity we have taken $f(1) = 0$ and $P, Q \in \Delta_n$.

Theorem 2.4. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex and normalized i.e., $f(1) = 0$. If $P, Q \in \Delta_n$ are such that*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, ..., n\},$$

for some r and R with $0 < r \leqslant 1 \leqslant R < \infty$, then we have the following inequalities:

$$0 \leqslant C_f(P||Q) \leqslant \frac{1}{4}(R-r)\left(f'(R) - f'(r)\right), \tag{2.5}$$

$$0 \leqslant C_f(P||Q) \leqslant \beta_f(r, R), \tag{2.6}$$

and

$$\begin{aligned} 0 &\leqslant \beta_f(r, R) - C_f(P||Q) \\ &\leqslant \frac{f'(R) - f'(r)}{R-r}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leqslant \frac{1}{4}(R-r)\left(f'(R) - f'(r)\right), \end{aligned} \tag{2.7}$$

where

$$\beta_f(r, R) = \frac{(R-1)f(r) + (1-r)f(R)}{R-r}, \tag{2.8}$$

and $\chi^2(P||Q)$ and $C_f(P||Q)$ are as given by (1.6) and (2.1) respectively.

In view of above theorem, we have the following result.

Result 2.5. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. If there exists r, R such that*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, ..., n\},$$

with $0 < r \leqslant 1 \leqslant R < \infty$, then we have

$$0 \leqslant \Phi_s(P||Q) \leqslant \mu_s(r, R), \tag{2.9}$$

$$0 \leqslant \Phi_s(P||Q) \leqslant \phi_s(r, R), \tag{2.10}$$

and

$$\begin{aligned} 0 &\leqslant \phi_s(r, R) - \Phi_s(P||Q) \\ &\leqslant k_s(r, R)\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leqslant \mu_s(r, R), \end{aligned} \tag{2.11}$$

where

$$\mu_s(r, R) = \begin{cases} \frac{1}{4}\frac{(R-r)\left(R^{s-1} - r^{s-1}\right)}{(s-1)}, & s \neq 1 \\ \frac{1}{4}(R-r)\ln\left(\frac{R}{r}\right), & s = 1 \end{cases}, \tag{2.12}$$

$$\begin{aligned} \phi_s(r, R) &= \frac{(R-1)\phi_s(r) + (1-r)\phi_s(R)}{R-r} \\ &= \begin{cases} \frac{(R-1)(r^s-1) + (1-r)(R^s-1)}{(R-r)s(s-1)}, & s \neq 0, 1 \\ \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{(R-r)}, & s = 0 \\ \frac{(R-1)r\ln r + (1-r)R\ln R}{(R-r)}, & s = 1 \end{cases}, \end{aligned} \tag{2.13}$$

and

$$k_s(r,R)=\frac{\phi_s'(R)-\phi_s'(r)}{R-r}=\begin{cases}\frac{R^{s-1}-r^{s-1}}{(R-r)(s-1)}, & s\neq 1\\ \frac{\ln R-\ln r}{R-r}, & s=1\end{cases}, \tag{2.14}$$

Proof. The above result follows immediately from Theorem 2.4, by taking $f(u)=\phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then in this case we have $C_f(P||Q)=\Phi_s(P||Q)$. □

Moreover,

$$\mu_s(r,R)=\frac{1}{4}(R-r)^2\,k_s(r,R),$$

where

$$k_s(r,R)=\begin{cases}[L_{s-2}(r,R)]^{s-2}, & s\neq 1\\ [L_{-1}(r,R)]^{-1} & s=1\end{cases},$$

and $L_p(a,b)$ is the famous (ref. Bullen, Mitrinović and Vasić [2]) *p-logarithmic mean* given by

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p\neq -1,0\\ \frac{b-a}{\ln b-\ln a}, & p=-1\\ \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}, & p=0\end{cases},$$

for all $p\in\mathbb{R}$, $a,b\in\mathbb{R}_+$, $a\neq b$.

We have the following corollaries as particular cases of Result 2.5.

Corollary 2.6. *Under the conditions of Result 2.5, we have*

$$0\leqslant K(Q||P)\leqslant\frac{(R-r)^2}{4Rr}. \tag{2.15}$$

$$0\leqslant K(P||Q)\leqslant\frac{1}{4}(R-r)\ln\left(\frac{R}{r}\right). \tag{2.16}$$

$$0\leqslant h(P||Q)\leqslant\frac{(R-r)\left(\sqrt{R}-\sqrt{r}\right)}{8\sqrt{Rr}}. \tag{2.17}$$

$$0\leqslant \chi^2(P||Q)\leqslant\frac{1}{2}(R-r)^2. \tag{2.18}$$

Proof. (2.15) follows by taking $s=0$, (2.16) follows by taking $s=1$, (2.17) follows by taking $s=\frac{1}{2}$ and (2.18) follows by taking $s=2$ in (2.9). □

Corollary 2.7. *Under the conditions of Result 2.5, we have*

$$0\leqslant K(Q||P)\leqslant\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}. \tag{2.19}$$

$$0\leqslant K(P||Q)\leqslant\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}. \tag{2.20}$$

$$0\leqslant h(P||Q)\leqslant\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}. \tag{2.21}$$

$$0\leqslant \chi^2(P||Q)\leqslant (R-1)(1-r). \tag{2.22}$$

Proof. (2.19) follows by taking $s=0$, (2.20) follows by taking $s=1$, (2.21) follows by taking $s=\frac{1}{2}$ and (2.22) follows by taking $s=2$ in (2.10). □

Corollary 2.8. *Under the conditions of Result 2.5, we have*

$$0 \leqslant \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \leqslant \frac{1}{rR}\left[(R-1)(1-r)-\chi^2(P||Q)\right]. \tag{2.23}$$

$$0 \leqslant \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r} - K(P||Q) \leqslant \frac{\ln R-\ln r}{R-r}\left[(R-1)(1-r)-\chi^2(P||Q)\right]. \tag{2.24}$$

$$0 \leqslant \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\left(\sqrt{R}+\sqrt{r}\right)} - h(P||Q) \leqslant \frac{2}{\sqrt{rR}\left(\sqrt{R}+\sqrt{r}\right)}\left[(R-1)(1-r)-\chi^2(P||Q)\right]. \tag{2.25}$$

Proof. (2.23) follows by taking $s=0$, (2.24) follows by taking $s=1$, (2.25) follows by taking $s=\frac{1}{2}$ in (2.11). □

3. Main Results

In this section, we shall present a theorem generalizing the one studied by Dragomir [8]. The results due to Dragomir [8] are limited only to χ^2- *divergence*, while the theorem studied here give in terms of *relative information of type s*, that in particular lead us to bounds in terms of χ^2-*divergence*, Kullback-Leibler's *relative information* and Hellinger's *discrimination.*

Theorem 3.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping is normalized, i.e., $f(1)=0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r,R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M with $m<M$ such that*

$$m \leqslant x^{2-s} f''(x) \leqslant M, \quad \forall x \in (r,R), \quad s \in \mathbb{R}. \tag{3.1}$$

If $P,Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$m\left[\phi_s(r,R)-\Phi_s(P||Q)\right] \leqslant \beta_f(r,R)-C_f(P||Q) \leqslant M\left[\phi_s(r,R)-\Phi_s(P||Q)\right], \tag{3.2}$$

where $C_f(P||Q), \Phi_s(P||Q), \beta_f(r,R)$ and $\phi_s(r,R)$ are as given by (2.1), (2.2), (2.8) and (2.13) respectively.

Proof. Let us consider the functions $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ given by

$$F_{m,s}(u) = f(u) - m\ \phi_s(u), \tag{3.3}$$

and

$$F_{M,s}(u) = M\phi_s(u) - f(u), \tag{3.4}$$

respectively, where m and M are as given by (3.1) and function $\phi_s(\cdot)$ is as given by (2.3).

Since $f(u)$ and $\phi_s(u)$ are normalized, then $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ are also normalized, i.e., $F_{m.s}(1) = 0$ and $F_{M.s}(1) = 0$. Moreover, the functions $f(u)$ and $\phi_s(u)$ are twice differentiable. Then in view of (2.4) and (3.1), we have

$$F''_{m,s}(u) = f''(u) - mu^{s-2} = u^{s-2}\left(u^{2-s}f''(u) - m\right) \geqslant 0$$

and

$$F''_{M,s}(u) = Mu^{s-2} - f''(u) = u^{s-2}\left(M - u^{2-s}f''(u)\right) \geqslant 0,$$

for all $u \in (r, R)$ and $s \in \mathbb{R}$. Thus the functions $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ are convex on (r, R).

We have seen above that the real mappings $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ defined over $\mathbb{R}_+$ given by (3.3) and (3.4) respectively are normalized, twice differentiable and convex on (r, R). Applying the *r.h.s.* of the inequality (2.6), we have

$$C_{F_{m,s}}(P||Q) \leqslant \beta_{F_{m,s}}(r, R), \tag{3.5}$$

and

$$C_{F_{m,s}}(P||Q) \leqslant \beta_{F_{M,s}}(r, R), \tag{3.6}$$

respectively.

Moreover,

$$C_{F_{m,s}}(P||Q) = C_f(P||Q) - m\ \Phi_s(P||Q), \tag{3.7}$$

and

$$C_{F_{M,s}}(P||Q) = M\ \Phi_s(P||Q) - C_f(P||Q), \tag{3.8}$$

In view of (3.5) and (3.7), we have

$$C_f(P||Q) - m\ \Phi_s(P||Q) \leqslant \beta_{F_{m,s}}(r, R),$$

i.e.,

$$C_f(P||Q) - m\ \Phi_s(P||Q) \leqslant \beta_f(r, R) - m\ \phi_s(r, R)$$

i.e.,

$$m\left[\phi_s(r, R) - \Phi_s(P||Q)\right] \leqslant \beta_f(r, R) - C_f(P||Q).$$

Thus, we have the *l.h.s.* of the inequality (3.2).

Again in view of (3.6) and (3.8), we have

$$M\ \Phi_s(P||Q) - C_f(P||Q) \leqslant \beta_{F_{M,s}}(r, R),$$

i.e.,

$$M\ \Phi_s(P||Q) - C_f(P||Q) \leqslant M\ \phi_s(r, R) - \beta_f(r, R),$$

i.e.,

$$\beta_f(r, R) - C_f(P||Q) \leqslant M\left[\phi_s(r, R) - \Phi_s(P||Q)\right].$$

Thus we have the *r.h.s.* of the inequality (3.2). □

We shall present now some particular case of the Theorem 3.1.

3.1. Information Bounds in Terms of χ^2-Divergence. In particular for $s = 2$, in Theorem 3.1, we have the following proposition:

Proposition 3.2. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping is normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r, R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M with $m < M$ such that*

$$m \leqslant f''(x) \leqslant M, \quad \forall x \in (r, R). \tag{3.9}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$\begin{aligned} &\frac{m}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leqslant \beta_f(r,R) - C_f(P||Q) \\ &\leqslant \frac{M}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{3.10}$$

where $C_f(P||Q), \beta_f(r,R)$ and $\chi^2(P||Q)$ are as given by (2.1), (2.8) and (1.6) respectively.

The above proposition is also studied by Dragomir [8]. As a consequence of Proposition 3.2, we have the following result.

Result 3.3. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exists r, R such that*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, ..., n\},$$

then in view of Theorem 3.1, we have

$$\begin{aligned} &\frac{R^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leqslant \phi_s(r,R) - \Phi_s(P||Q) \\ &\leqslant \frac{r^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \ s \leqslant 2. \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\frac{r^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leqslant \phi_s(r,R) - \Phi_s(P||Q) \\ &\leqslant \frac{R^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \ s \geqslant 2. \end{aligned} \tag{3.12}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Now if $u \in [r, R] \subset (0, \infty)$, then we have

$$R^{s-2} \leqslant \phi_s''(u) \leqslant r^{s-2}, \quad s \leqslant 2,$$

or accordingly, we have

$$\phi_s''(u) \begin{cases} \leqslant r^{s-2}, & s \leqslant 2 \\ \geqslant r^{s-2}, & s \geqslant 2 \end{cases} \tag{3.13}$$

and

$$\phi_s''(u) \begin{cases} \leqslant R^{s-2}, & s \geqslant 2 \\ \geqslant R^{s-2}, & s \leqslant 2 \end{cases} \tag{3.14}$$

where r and R are as defined above. Thus in view of (3.9), (3.13) and (3.14), we have the proof. □

In view of Result 3.3, we have the following corollary.

Corollary 3.4. *Under the conditions of Result 3.3, we have*

$$\begin{aligned} \frac{1}{2R^2} & \left[(R-1)(1-r) - \chi^2(P\|Q)\right] \\ & \leqslant \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q\|P) \\ & \leqslant \frac{1}{2r^2}\left[(R-1)(1-r) - \chi^2(P\|Q)\right]. \end{aligned} \tag{3.15}$$

$$\begin{aligned} \frac{1}{2R} & \left[(R-1)(1-r) - \chi^2(P\|Q)\right] \\ & \leqslant \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P\|Q) \\ & \leqslant \frac{1}{2r}\left[(R-1)(1-r) - \chi^2(P\|Q)\right]. \end{aligned} \tag{3.16}$$

$$\begin{aligned} \frac{1}{2\sqrt{R^3}} & \left[(R-1)(1-r) - \chi^2(P\|Q)\right] \\ & \leqslant \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P\|Q) \\ & \leqslant \frac{1}{2\sqrt{r^3}}\left[(R-1)(1-r) - \chi^2(P\|Q)\right]. \end{aligned} \tag{3.17}$$

Proof. (3.15) follows by taking $s = 0$, (3.16) follows by taking $s = 1$, (3.17) follows by taking $s = \frac{1}{2}$ in Result 3.3. While for $s = 2$, we have equality sign. □

3.2. Information Bounds in Terms of Kullback-Leibler Relative Information. In particular for $s=1$, in the Theorem 3.1, we have the following proposition.

Proposition 3.5. *Let* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *the generating mapping is normalized, i.e.,* $f(1)=0$ *and satisfy the assumptions:*

(i) f *is twice differentiable on* (r,R) , *where* $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;
(ii) *there exists the real constants* m, M *with* $m < M$ *such that*

$$m \leqslant x f''(x) \leqslant M, \quad \forall x \in (r,R). \tag{3.18}$$

If $P, Q \in \Delta_n$ *are discrete probability distributions satisfying the assumption*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$\begin{aligned} & m\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ & \leqslant \beta_f(r,R) - C_f(P||Q) \\ & \leqslant M\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \end{aligned} \tag{3.19}$$

where $C_f(P||Q), \beta_f(r,R)$ *and* $K(P||Q)$ *as given by (2.1), (2.8) and (1.1) respectively.*

In view of above proposition, we have the following result.

Result 3.6. *Let* $P, Q \in \Delta_n$ *and* $s \in \mathbb{R}$. *Let there exists* r, R *such that*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1,2,...,n\},$$

then in view of Proposition 3.5, we have

$$\begin{aligned} & r^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ & \leqslant \phi_s(r,R) - \Phi_s(P||Q) \\ & \leqslant R^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \ s \geqslant 1. \end{aligned} \tag{3.20}$$

$$\begin{aligned} & R^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ & \leqslant \phi_s(r,R) - \Phi_s(P||Q) \\ & \leqslant r^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \ s \leqslant 1. \end{aligned} \tag{3.21}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \to \mathbb{R}$ such that $g(u) = ug''(u) = u^{s-1}$, then we have

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^{s-1}, & s \geqslant 1 \\ r^{s-1}, & s \leqslant 1 \end{cases} \tag{3.22}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{s-1}, & s \geqslant 1 \\ R^{s-1}, & s \leqslant 1 \end{cases} \tag{3.23}$$

In view of (3.22), (3.23) and Proposition 3.5 we have the proof of the result. □

In view of Result 3.6, we have the following corollary.

Corollary 3.7. *Under the conditions of Result 3.6, we have*

$$\begin{aligned} &\frac{1}{R}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ &\leqslant \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\ &\leqslant \frac{1}{r}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right]. \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\frac{1}{\sqrt{R}}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ &\leqslant \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\ &\leqslant \frac{1}{\sqrt{r}}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right]. \end{aligned} \tag{3.25}$$

$$\begin{aligned} &r\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ &\leqslant (R-1)(1-r) - \chi^2(P||Q) \\ &\leqslant R\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right]. \end{aligned} \tag{3.26}$$

Proof. Part (i) follows by taking $s = 0$, part (ii) follows by taking $s = \frac{1}{2}$ and part (iii) follows by taking $s = 2$ in Result 3.6. For $s = 1$, we have equality sign. □

In particular for $s = 0$, in the Theorem 3.1, we have the following proposition:

Proposition 3.8. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping is normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r,R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M with $m < M$ such that*

$$m \leqslant x^2 f''(x) \leqslant M, \quad \forall x \in (r,R). \tag{3.27}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$\begin{aligned} & m\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] \\ & \leqslant \beta_f(r,R) - C_f(P||Q) \\ & \leqslant M\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \end{aligned} \tag{3.28}$$

where $C_f(P||Q), \beta_f(r,R)$ and $K(Q||P)$ as given by (2.1), (2.8) and (1.1) respectively.

In view of Proposition 3.8, we have the following result.

Result 3.9. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exists r, R such that*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1,2,...,n\}$$

then in view of Proposition 3.8, we have

$$\begin{aligned} & r^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] \\ & \leqslant \phi_s(r,R) - \Phi_s(P||Q) \\ & \leqslant R^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \ s \geqslant 0, \end{aligned} \tag{3.29}$$

$$\begin{aligned} & R^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] \\ & \leqslant \phi_s(r,R) - \Phi_s(P||Q) \\ & \leqslant r^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \ s \leqslant 0. \end{aligned} \tag{3.30}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r,R] \to \mathbb{R}$ such that $g(u) = u^2 g''(u) = u^s$, then we have

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^s, & s \geqslant 0 \\ r^s, & s \leqslant 0 \end{cases} \tag{3.31}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^s, & s \geqslant 0 \\ R^s, & s \leqslant 0 \end{cases} \tag{3.32}$$

In view of (3.31), (3.32) and Proposition 3.8, we have the proof of the result. ☐

In view of Result 3.9, we have the following corollaries.

Corollary 3.10. *Under the conditions of Result 3.9, we have*

$$\begin{aligned} r\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] &\leqslant \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r} - K(P||Q) \\ &\leqslant R\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \end{aligned} \tag{3.33}$$

$$\begin{aligned} \sqrt{r}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] &\leqslant \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\ &\leqslant \sqrt{R}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \end{aligned} \tag{3.34}$$

$$\begin{aligned} r^2\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] &\leqslant (R-1)(1-r) - \chi^2(P||Q) \\ &\leqslant R^2\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right]. \end{aligned} \tag{3.35}$$

Proof. (3.33) follows by taking $s = 1$, (3.34) follows by taking $s = \frac{1}{2}$ and 3.35) follows by taking $s = 2$ in Result 3.9. For $s = 0$, we have equality sign. ☐

3.3. Information Bounds in Terms of Hellinger's Discrimination.

In particular for $s = \frac{1}{2}$, in Theorem 3.1, we have the following proposition:

Proposition 3.11. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping is normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r, R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M with $m < M$ such that*

$$m \leqslant x^{3/2} f''(x) \leqslant M, \quad \forall x \in (r, R). \tag{3.36}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$4m\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]\leqslant\beta_f(r,R)-C_f(P||Q)\leqslant 4M\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right],\tag{3.37}$$

where $C_f(P||Q), \beta_f(r,R)$ and $h(P||Q)$ as given by (2.1), (2.8) and (1.5) respectively.

In view of Proposition 3.11, we have the following result.

Result 3.12. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exists r, R such that*

$$0<r\leqslant\frac{p_i}{q_i}\leqslant R<\infty,\quad \forall i\in\{1,2,...,n\},$$

then in view of Proposition 3.11, we have

$$\begin{aligned}&4r^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]\\&\leqslant\phi_s(r,R)-\Phi_s(P||Q)\\&\leqslant 4R^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right],\ s\geqslant\frac{1}{2}.\end{aligned}\tag{3.38}$$

$$\begin{aligned}&4R^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]\\&\leqslant\phi_s(r,R)-\Phi_s(P||Q)\\&\leqslant 4r^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right],\ s\leqslant\frac{1}{2}.\end{aligned}\tag{3.39}$$

Proof. Let the function $\phi_s(u)$ given by (3.18) is defined over $[r,R]$. Defining $g(u)=u^{3/2}\phi_s''(u)=u^{3/2}u^{s-2}=u^{\frac{2s-1}{2}}$, obviously we have

$$\sup_{u\in[r,R]}g(u)=\begin{cases}R^{\frac{2s-1}{2}}, & s\geqslant\frac{1}{2}\\ r^{\frac{2s-1}{2}}, & s\leqslant\frac{1}{2}\end{cases}\tag{3.40}$$

and

$$\inf_{u\in[r,R]}g(u)=\begin{cases}r^{\frac{2s-1}{2}}, & s\geqslant\frac{1}{2}\\ R^{\frac{2s-1}{2}}, & s\leqslant\frac{1}{2}\end{cases}\tag{3.41}$$

In view of (3.40), (3.41) and Proposition 3.11, we get the proof of the result. □

In view of Result 3.12, we have the following corollary.

Corollary 3.13. *Under the conditions of Result 3.12, we have*

(3.42)
$$\frac{4}{\sqrt{R}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \leqslant \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P) \leqslant \frac{4}{\sqrt{r}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right].$$

(3.43)
$$4\sqrt{r}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \leqslant \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q) \leqslant 4\sqrt{R}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right].$$

(3.44)
$$8\sqrt{r^3}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \leqslant (R-1)(1-r)-\chi^2(P||Q) \leqslant 8\sqrt{R^3}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right].$$

Proof. (3.42) follows by taking $s=0$, (3.43) follows by taking $s=1$ and (3.44) follows by taking $s=2$ in Result 3.12. For $s=\frac{1}{2}$, we have equality sign. □

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