A NOTE ON HARDY-TYPE INEQUALITIES

PENG GAO

Abstract. We use a theorem of Cartlidge and the technique of Redheffer's "recurrent inequalities" to give some results on inequalities related to Hardy's inequality.

1. Introduction

Suppose throughout that $p \neq 0, \frac{1}{p}+\frac{1}{q}=1$. Let l^p be the Banach space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$ with norm

$$||\mathbf{a}|| := (\sum_{n=1}^{\infty}|a_n|^p)^{1/p} < \infty.$$

The celebrated Hardy's inequality([10], Theorem 326) asserts that for $p>1$,

$$\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}a_k)^p \leq (\frac{p}{p-1})^p\sum_{k=1}^{\infty}|a_k|^p. \tag{1.1}$$

Among the many papers appeared providing new proofs, generalizations and sharpenings of (1.1), we refer the reader to the work of G.Bennett [2]-[6] for his study of factorable matrices.

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{j=1}^{\infty}\Big|\sum_{k=1}^{\infty}c_{j,k}a_k\Big|^p \leq U\sum_{k=1}^{\infty}|a_k|^p,$$

in which $C=(c_{j,k})$ and the parameter p are assumed fixed($p>1$), and the estimate is to hold for all real sequences $\mathbf{a}$. The l^p *operator norm* of C is then defined as the p-th root of the smallest value of the constant U:

$$||C||_{p,p}=U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the *Cesáro matrix operator* C, given by $c_{j,k}=1/j, k\leq j$ and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

We say a matrix A is a summability matrix if its entries satisfy: $a_{j,k}\geq 0$, $a_{j,k}=0$ for $k>j$ and $\sum_{k=1}^{j}a_{j,k}=1$. We say a summability matrix A is a weighted mean matrix if its entries satisfy:

$$a_{j,k}=\lambda_k/\Lambda_j,\ 1\leq k\leq j; \Lambda_j=\sum_{i=1}^{j}\lambda_i. \tag{1.2}$$

We refer to the n-tuple $(a_{n1},a_{n2},\cdots,a_{nn})$ as the n-th row of a summability matrix A and then have the following result of Bennett([6], Theorem 1.14) for the l^p *operator norm* of A.

Theorem 1.1. *Let $p>1$ be fixed and suppose A is a summability matrix. If the rows of A are decreasing, then $||A||_{p,p}\geq p/(p-1)$. If the rows of A are increasing, then $||A||_{p,p}\leq p/(p-1)$.*

The above theorem, when applied to weighted mean matrixes, gives the following inequality([6], Corollary 4.10).

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Theorem 1.2. *If $0<\lambda_1\leq\lambda_2\leq\cdots$ and $0<p<1$, then*

$$\sum_{n=1}^{\infty}\Big(\frac{\sum_{i=1}^{n}\lambda_i a_i^p}{\sum_{i=1}^{n}\lambda_i}\Big)^{1/p}\leq\Big(\frac{1}{1-p}\Big)^{1/p}\sum_{n=1}^{\infty}a_n, \tag{1.3}$$

whenever **a** *is a sequence of non-negative terms.*

Even though the constant in the above theorem is best possible, some improvement may be possible with specific choices of the λ_i's. For examples, the following two inequalities were claimed to hold by Bennett([5], page 40-41; see also [6], page 407):

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{i=1}^{n}(i^{\alpha}-(i-1)^{\alpha})a_i\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p, \tag{1.4}$$

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\sum_{i=1}^{n}i^{\alpha-1}}\sum_{i=1}^{n}i^{\alpha-1}a_i\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p, \tag{1.5}$$

whenever $\alpha>0, p>1, \alpha p>1$.

We haven't seen the proofs of Bennett but find the following unpublished result of J. Cartlidge[7] is very helpful to treat the above two inequalities. We don't have access to his thesis either, so here we quote the one in [2](p. 416):

Theorem 1.3. *Let $1<p<\infty$ be fixed. Let A be a weighted mean matrix given by* (1.2). *If*

$$L=\sup_n\Big(\frac{\Lambda_{n+1}}{\lambda_{n+1}}-\frac{\Lambda_n}{\lambda_n}\Big)<p\ , \tag{1.6}$$

then $||A||_{p,p}\leq p/(p-L)$.

We will apply the above theorem to prove (1.4)-(1.5) for $\alpha\geq 2, p>1, \alpha p>1$ in section 3.

Suppose $a_n\geq 0$, by a change of variables $a_n\to a_n^{1/p}$ and let $p\to\infty$, (1.1) gives the well-known Carleman's inequality:

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n}a_k\Big)^{\frac{1}{n}}\leq e\sum_{n=1}^{\infty}a_n.$$

We refer the reader to the survey article [13] and the references therein for an account of Carleman's inequality. Among the various generalizations of Carleman's inequality, we mention a result of E. Love, who proved for $\alpha>0, \lambda_i=i^{\alpha}-(i-1)^{\alpha}$,

$$\sum_{n=1}^{\infty}\Big(\prod_{i=1}^{n}a_i^{i^{\alpha}-(i-1)^{\alpha}}\Big)^{1/n^{\alpha}}\leq e^{\frac{1}{\alpha}}\sum_{n=1}^{\infty}a_n, \tag{1.7}$$

and the constant $e^{\frac{1}{\alpha}}$ is best possible. We note here after a change of variables $a_n\to a_n^{1/p}$, (1.7) corresponds to the limiting case $p\to\infty$ of (1.4).

R.Redheffer gave a remarkable proof of Hardy's inequality in [14] by developing the method of "recurrent inequalities". His method also works for Carleman's inequality. Another proof of Carleman's inequality was given by him in [15] and his result has been generalized by H.Alzer[1] and most recently by J. Pečarić and K. Stolarsky[13], who proved for $b_n>0$, $N\geq 1, G_n=(\prod_{i=1}^{n}a_i)^{1/n}$,

$$\sum_{n=1}^{N}\Lambda_n(b_n-1)G_n+\Lambda_N G_N\leq\sum_{n=1}^{N}\lambda_n G_n b_n^{\Lambda_n/\lambda_n}.$$

In this paper, we will use Redheffer's method to give a weighted version of his treatment of Hardy's and Carleman's inequalities. As we shall see, our result for $1<p<\infty$ is less satisfactory than that of Cartlidge's while for the limiting case the result is almost the same as his.

From now on we will assume $a_n\geq 0$ for $n\geq 1$ and any infinite sum converges.

2. Lemmas

Lemma 2.1. *Let* $\Lambda_k = \sum_{i=1}^{k}\lambda_i$, $\lambda_i > 0$ *and* $S_n = \sum_{i=1}^{n}\lambda_i a_i$. *Let* $0 \neq p < 1$ *be fixed and let* $(\mu_n)_{n\geq 1}, (\eta_n)_{n\geq 1}$ *be two sequences of real numbers such that* $\mu_i \leq \eta_i$ *for* $0 < p < 1$ *and* $\mu_i \geq \eta_i$ *for* $p < 0$, *then for* $n \geq 2$,

$$\sum_{i=2}^{n-1}[\mu_i - (\mu_{i+1}^q - \eta_{i+1}^q)^{1/q}]S_i^{1/p} + \mu_n S_n^{1/p} \leq (\mu_2^q - \eta_2^q)^{1/q}\lambda_1^{1/p}a_1^{1/p} + \sum_{i=2}^{n}\eta_i\lambda_i^{1/p}a_i^{1/p}. \tag{2.1}$$

Proof. This is essentially due to R.Redheffer[14]. We note for $k \geq 2$,

$$\mu_k S_k^{1/p} - \eta_k\lambda_k^{1/p}a_k^{1/p} = S_{k-1}^{1/p}(\mu_k(1+t)^{1/p} - \eta_i t^{1/p}) \leq (\mu_k^q - \eta_k^q)^{1/q}S_{k-1}^{1/p}, \tag{2.2}$$

with $t = \lambda_k a_k/S_{k-1}$(compare this with the one on page 688 of [14]). The lemma then follows by adding (2.2) for $2 \leq k \leq n$ together. □

Lemma 2.2. *Let* $\Lambda_k = \sum_{i=1}^{k}\lambda_i$, $\lambda_i > 0$ *and* $G_k = (\prod_{i=1}^{k}a_i^{\lambda_i})^{1/\Lambda_k}$, *then for* $\mu_i > 0, n \geq 2$,

$$G_1 + \sum_{i=2}^{n-1}(\frac{\Lambda_i\mu_i}{\lambda_i} - \frac{\Lambda_i}{\lambda_{i+1}})G_i + \frac{\Lambda_n\mu_n}{\lambda_n}G_n \leq (1 + \frac{\Lambda_1}{\lambda_2})a_1 + \sum_{i=2}^{n}\mu_i^{\frac{\Lambda_i}{\lambda_i}}a_i. \tag{2.3}$$

Proof. This is essentially due to R.Redheffer[14]. We note for $k \geq 2, \mu > 0, \eta > 0$,

$$\mu G_k - \eta a_k = G_{k-1}(\mu t - \eta t^{\frac{\Lambda_k}{\lambda_k}}) \leq G_{k-1}(\frac{\Lambda_{k-1}}{\lambda_k})\eta^{\frac{-\lambda_k}{\Lambda_{k-1}}}(\frac{\mu\lambda_k}{\Lambda_k})^{\frac{\Lambda_k}{\Lambda_{k-1}}},$$

where $t^{\frac{\Lambda_k}{\lambda_k}} = a_k/G_{k-1}$(compare this with the one on page 686 of [14]). By setting $\mu_k\Lambda_k/\lambda_k = \mu, \eta_k = \eta = \mu_k^{\Lambda_k/\lambda_k}$, we get

$$\frac{\Lambda_k\mu_k}{\lambda_k}G_k - a_k\mu_k^{\frac{\Lambda_k}{\lambda_k}} \leq \frac{\Lambda_{k-1}}{\lambda_k}G_k. \tag{2.4}$$

The lemma then follows by adding (2.4) for $2 \leq k \leq n$ and $G_1 = a_1$ together. □

Lemma 2.3. *Let* $f(x) \in C^3[a,b]$ *and* $f'''(x) \geq 0$ *for* $x \in [a,b]$. *Then*

$$f(b) - f(a) \geq f'(\frac{a+b}{2})(b-a). \tag{2.5}$$

Proof. By Taylor's expansion,

$$\begin{aligned} f(b) &= f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(b - \frac{a+b}{2}) + f''(\eta_1)(a-b)^2/4, \\ f(a) &= f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(a - \frac{a+b}{2}) + f''(\eta_2)(a-b)^2/4, \end{aligned}$$

where $a < \eta_2 < (a+b)/2 < \eta_1 < b$. The lemma then follows by noticing $f'''(x) \geq 0$ for $x \in [a,b]$. □

Lemma 2.4. *If* $s \geq 1$, *then*

$$\sum_{i=1}^{n} i^s \geq \frac{s}{s+1}\frac{n^s(n+1)^s}{(n+1)^s - n^s}. \tag{2.6}$$

Proof. This is a result of V. Levin and S. Stečkin, see Lemma 2 on page 18 in [11]. □

3. Applications of Cartlidge's Theorem

We say a weighted mean matrix A given by (1.2) is generated by a logarithmico-exponential function if for all sufficiently large n, $\lambda_n := l(n)$, where $l(x)$ is a positive logarithmico-exponential function and a logarithmico-exponential function on $[x_0, \infty]$ is defined by Hardy[9] as a real valued function defined by a finite combination of ordinary algebraic symbols(viz, $+, -, \times, \div, \sqrt[n]{}$) and the functional symbols $log(\cdot)$ and $e^{(\cdot)}$, operating on real variable x and on real constants.

We note first the following theorem of F. Cass and W. Kratz[8]:

Theorem 3.1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.2)*. Suppose $\lim_{n\to\infty} \Lambda_n/n\lambda_n = L < p$, then $p/(p-L) \leq ||A||_{p,p}$.*

It is easy to see $\lim_{n\to\infty} n^{\alpha-1}/(n^\alpha - (n-1)^\alpha) = 1/\alpha$ and the simplest Euler-Maclaurin formulae gives:

$$\sum_{i=1}^n f(i) = \int_1^n f(x)dx + f(1) + \int_1^n (x-[x])f'(x)dx,$$

for f having continuous derivative f', where $[x]$ denote the largest integer not exceeding the real number x. It then follows

$$\sum_{i=1}^n i^{\alpha-1} = n^\alpha/\alpha + o(n^\alpha).$$

Thus thanks to Theorem 3.1, we know if (1.4)-(1.5) hold for some $\alpha > 0, p > 1, \alpha p > 1$ then the constants $(\alpha p/(\alpha p-1))^p$ are best possible.

Now we apply Cartlidge's Theorem to get

Corollary 3.1. *Inequality* (1.4) *holds for $p > 1, \alpha \geq 2, \alpha p > 1$ and the constant there is best possible.*

Proof. Apply Theorem 1.3 with $\lambda_i = i^\alpha - (i-1)^\alpha$. We define $f(x) = x^\alpha/(x^\alpha - (x-1)^\alpha), x \geq 1$ so that $\Lambda_{i+1}/\lambda_{i+1} - \Lambda_i/\lambda_i = f(i+1) - f(i) = f'(\xi), 1 \leq i < \xi < i+1$, with

$$0 < f'(\xi) = \frac{\alpha\xi^{\alpha-1}(\xi-1)^{\alpha-1}}{(\xi^\alpha - (\xi-1)^\alpha)^2} \leq \frac{1}{\alpha},$$

where the last inequality follows from Lemma 2.3 and the arithmetic-geometric inequality, since for $\alpha \geq 2$,

$$\xi^\alpha - (\xi-1)^\alpha \geq \alpha(\frac{\xi + (\xi-1)}{2})^{\alpha-1} \geq \alpha(\xi(\xi-1))^{(\alpha-1)/2}.$$

This completes the proof. □

We note the corollary implies (1.7) for $\alpha \geq 2$. Now if we apply Theorem 1.3 to (1.5), we need to show

$$\sum_{i=1}^{n+1} i^{\alpha-1}/(n+1)^{\alpha-1} - \sum_{i=1}^n i^{\alpha-1}/n^{\alpha-1} = 1 + (\frac{1}{(n+1)^{\alpha-1}} - \frac{1}{n^{\alpha-1}})\sum_{i=1}^n i^{\alpha-1} \leq 1/\alpha.$$

The second inequality above follows from Lemma 2.4 and we get

Corollary 3.2. *Inequality* (1.5) *holds for $p > 1, \alpha \geq 2, \alpha p > 1$ and the constant there is best possible.*

4. Generalizations of Redheffer's Results

Theorem 4.1. *Assume the same conditions in Lemma 2.1 and let* $0 < p < 1$ *be fixed. Suppose there exists a positive constant* c *such that* $c^{-1} + 1 \leq c^{-1/p}$ *and*

$$c \leq 1 - p + (1-p)(\lambda_i^{-q} - \lambda_{i-1}^{-q})\Lambda_{i-1}\lambda_i^{q/p}, i \geq 2. \tag{4.1}$$

Then for $0 < p < 1$,

$$\sum_{i=1}^{\infty}(S_i/\Lambda_i)^{1/p} \leq c^{-1/p}\sum_{i=1}^{\infty} a_i^{1/p}. \tag{4.2}$$

Proof. It suffices to prove the theorem for any integer $n \geq 1$. We note first the condition (4.1) is equivalent to

$$q^{-1}(1 - c^{-1} + c^{-1}\Lambda_{i-1}\lambda_i^{q/p}(\lambda_{i-1}^{-q} - \lambda_i^{-q})) \geq 1, i \geq 2. \tag{4.3}$$

By setting $\eta_i = \lambda_i^{-1/p}, \mu_i^q = \lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q$ in (2.1), we can rewrite the left-hand side of (2.1) as

$$(1 - c^{-1/q})a_1^{1/p} + \sum_{i=2}^{n-1}[(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c\lambda_i^q)^{1/q}]S_i^{1/p} + \mu_n S_n^{1/p}.$$

By the mean value theorem,

$$\begin{aligned}(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c\lambda_i^q)^{1/q} &\geq q^{-1}(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q - \Lambda_i/c\lambda_i^q)(\Lambda_i/c\lambda_i^q)^{-1/p} \\ &= q^{-1}(1 - c^{-1} + c^{-1}\Lambda_{i-1}\lambda_i^{q/p}(\lambda_{i-1}^{-q} - \lambda_i^{-q}))(\Lambda_i/c)^{-1/p} \\ &\geq (\Lambda_i/c)^{-1/p}.\end{aligned}$$

Here the last inequality follows from (4.3). Thus (2.1) becomes

$$\sum_{i=1}^{n}(S_i/\lambda_i)^{1/p} \leq (c^{-1}+1)a_1 + c^{-1/p}\sum_{i=2}^{n} a_i \leq c^{-1/p}\sum_{i=1}^{n} a_i.$$

This completes the proof. □

We note here if $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, we can take $c = 1 - p$ in (4.1) and one checks easily for $0 < p < 1$, $(1-p)^{-1} + 1 < (1-p)^{-1/p}$. Theorem 4.1 then implies Theorem 1.2.

We also note the constant given by the above theorem may be less satisfactory. For example the case $\alpha = 2, p = 2$ in (1.4) corresponds to the case $\lambda_i = 2i - 1, p = 1/2, c = 3/4$ in (4.2). However, direct calculation shows (4.1) is not satisfied in this case. Of course one may try to prove directly

$$(\lambda_i^{-q/p} + \Lambda_{i-1}/c\lambda_{i-1}^q)^{1/q} - (\Lambda_i/c\lambda_i^q)^{1/q} \geq (\Lambda_i/c)^{-1/p}.$$

But one checks this fails for $i = 2$.

Similarly, the case $\alpha = 2, p = 2$ in (1.5) corresponds to the case $\lambda_i = i, p = 1/2, c = 3/4$ in (4.2). One checks in this case (4) holds for $i \geq 2$. However, $c^{-1} + 1 = 7/3 > 16/9 = c^{-2}$, so the coefficient of a_1 is slightly larger.

Now we focus our attention to Carleman-type inequalities.

Theorem 4.2. *Assume the same conditions in Lemma 2.2 and let* $f(x)$ *be a real valued function defined for* $x \geq 2$ *such that* $f(n) = \Lambda_n/\lambda_n$ *for* $n \geq 2$ *and* $0 \leq f(x+1) - f(x) \leq 1/\alpha$ *for some* $\alpha > 0$. *If* $(1 + \frac{\Lambda_1}{\lambda_2}) \leq e^{1/\alpha}$ *for the same* α *then*

$$\sum_{n=1}^{\infty}(\prod_{i=1}^{n} a_i^{\lambda_i})^{1/\Lambda_n} \leq (1 + \frac{\Lambda_1}{\lambda_2})a_1 + \sum_{i=2}^{n} a_i(1 + \frac{f(i+1) - f(i)}{f(i)})^{f(i)} \leq e^{1/\alpha}\sum_{n=1}^{\infty} a_n. \tag{4.4}$$

Proof. It suffices to prove the theorem for any integer $n \geq 2$. Set $\mu_i = f(i+1)/f(i)$ in Lemma 2.2 we get

$$\sum_{i=1}^{n} G_i \leq \sum_{i=1}^{n-1} G_i + f(n+1)G_N \leq (1+\frac{\Lambda_1}{\lambda_2})a_1 + \sum_{i=2}^{n} a_i(1+\frac{f(i+1)-f(i)}{f(i)})^{f(i)} \leq e^{1/\alpha}\sum_{n=1}^{\infty} a_n,$$

by the conditions of the theorem and this completes the proof. □

Apply Theorem 4.2 to $\lambda_1 = 1, \lambda_i = \alpha^{i-1} - \alpha^{i-2}, i \geq 2$ for some $\alpha > 1$, then $f(x) = \alpha/(\alpha-1)$ and we get

Corollary 4.1. *For* $\alpha > 1$*,*

$$\sum_{n=1}^{\infty}(a_1\prod_{k=2}^{n} a_k^{\alpha^{k-1}-\alpha^{k-2}})^{1/\alpha^{n-1}} \leq (1+\frac{1}{\alpha-1})a_1 + \sum_{n=2}^{\infty} a_n. \tag{4.5}$$

Apply Theorem 4.2 to $\lambda_i = \alpha^i, i \geq 1$ for some $\alpha > 0$, then $f(i+1) - f(i) = \alpha^{-i}$ and we get

Corollary 4.2. *For* $\alpha > 0$*,*

$$\sum_{n=1}^{\infty}(\prod_{k=1}^{n} a_k^{\alpha^{k-1}})^{(\alpha^n-1)/(\alpha-1)} \leq (1+\frac{1}{\alpha})a_1 + \sum_{n=2}^{\infty} e^{1/\alpha^n}a_n \leq \sum_{n=1}^{\infty} e^{1/\alpha^n}a_n. \tag{4.6}$$

We end the paper by noting that if we take $\lambda_i = (i(i+1))^{-1}$ in Theorem 4.2, then $f(x) = x^2$ and we get back a result of Redheffer(see [14]page 693):

Corollary 4.3.

$$\sum_{n=1}^{\infty}(\prod_{k=1}^{n} a^{1/k(k+1)})^{(n+1)/n} \leq \sum_{n=1}^{\infty} e^{2n}a_n. \tag{4.7}$$

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: penggao@umich.edu