ON MATHIEU-BERG'S INEQUALITY

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ABSTRACT. In this paper, by using Euler-Maclaurin's summation formula, we give a new improvement of Mathieu-Berg's inequality.
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1. INTRODUCTION

Let s(c) be the Mathieu's series as(see [1])

$$s(c) := \sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} \ (c \in R). \tag{1.1}$$

The inequality of the form

$$\frac{1}{c^2+1/2} < s(c) < \frac{1}{c^2} \ (c > 0) \tag{1.2}$$

is usually called Mathieu-Berg's inequality, which was guessed by E. mathieu in 1890 and proved by L. Berg in 1952 (see[2]). Since then, a good few mathematicians have had various refinements of it. The representative work of them is the result of the form as(see[3]):

$$s(c) = \sum_{i=0}^{k-1} (-1)^i \frac{B_{2i}}{c^{2i+2}} + \delta_k(c) \ (c > 0), \tag{1.3}$$

where $\delta_k(c) = (-1)^k \frac{2^{2k}-1}{2^{2k}} \left[1 + \theta_k \frac{2^{2k}+1}{2^{2k}} \binom{2k}{k}\right] \frac{B_{2k}}{c^{2k+2}} \ (|\theta_k| < 1), B_k's$ are Bernoulli's numbers, $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \cdots$. In particular, for k=1,2, reducing (1.3), we have

$$\frac{1}{c^2} - \frac{1}{6c^4} - \frac{23}{256c^6} < s(c) < \frac{1}{c^2} - \frac{1}{6c^4} + \frac{7}{256c^6}; \tag{1.4}$$

$$\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} - \frac{153}{2048c^8} < s(c) < \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8}. \tag{1.5}$$

For the remainder term $\delta_k(c)$, D.C. Russell [4] gave the following integral form:

$$\delta_k(c) = \frac{(-1)^{k-1}}{c^{2k}} \int_0^{\infty} \big(\frac{x}{e^x-1}\big)^{2k-1} cos(cx)dx. \tag{1.6}$$

In this paper, by using (1.4),(1.5)and Euler-Maclaurin's summation formula, a sharp result of Mathieu-Berg's inequality is obtained which is an improvement of (1.2).

2. SOME LEMMAS

Lemma 2.1. If $f(x) \in C^2[2,\infty), f'(x) < 0$, and $f(\infty) = 0$, then we have

$$0 < \int_2^\infty P_3(x)f(x)dx < \frac{1}{64}f(2), \tag{2.1}$$

where $P_j(x)$ $(j = 1, 2, \cdots)$ are Bernoulli's function(see[5]).

Proof. Basing on the periodicity of Bernoulli's function and observing that $|B_4| = max|P_4(x)|$, we have

$$\int_2^\infty \big(|B_4| - |P_4(x)|\big)|f'(x)|dx > 0.$$

Since $f'(x) < 0$, and $f(\infty) = 0$, then $f(x) \downarrow 0$ $(x \to \infty)$, and $f(x) > 0$. Since $B_4 < 0$, we obtain

$$\int_2^\infty P_4(x)f'(x)dx \le \int_2^\infty |P_4(x)||f'(x)|dx < |B_4|\int_2^\infty |f'(x)|dx = -B_4 f(2).$$

By using the relation $P_4'(x) = 4P_3(x)$, and $P_4(2) = B_4$, we have

$$\int_2^\infty P_3(x)f(x)dx = \frac{1}{4}\int_2^\infty f(x)dP_4(x) = \frac{1}{4}\big[-B_4 f(2) - \int_2^\infty P_4(x)f'(x)dx\big] > 0. \tag{2.2}$$

In view of that fact that(see[1, p.83])

$$P_3(x) = \sum_{v=0}^{3}\binom{3}{v}B_v x^{3-v} \ (x \in [0,1]), and \int_k^{k+1} P_3(x)dx = 0 \ (k = 1, 2, \cdots),$$

we obtain

$$\int_2^\infty P_3(x)f(x)dx = \sum_{k=2}^\infty \int_k^{k+1} P_3(x)(f(x) - f(k+1))dx$$

$$= \sum_{k=2}^\infty \Big\{\int_k^{k+1/2} P_3(x)(f(x) - f(k+1))dx + \int_{k+1/2}^{k+1} P_3(x)(f(x) - f(k+1))dx\Big\}$$

$$= \sum_{k=2}^\infty (f(k) - f(k+1))\int_k^{k+1/2} P_3(x)dx + \sum_{k=2}^\infty \alpha_k$$

$$= \sum_{k=2}^\infty (f(k) - f(k+1))\int_0^{1/2} P_3(x)dx + \sum_{k=2}^\infty \alpha_k$$

$$= f(2)\sum_{v=0}^{3}\binom{3}{v}B_v \int_0^{1/2} x^{3-v}dx + \sum_{k=2}^\infty \alpha_k = \frac{1}{64}f(2) + \sum_{k=2}^\infty \alpha_k,$$

where α_k is defined by

$$\alpha_k := \int_k^{k+1/2} P_3(x)(f(x) - f(k))dx - \int_{k+1/2}^{k+1} P_3(x)(f(k+1) - f(x))dx.$$

Note that $f(x) \downarrow 0$ $(x \to \infty)$, we have $\alpha_k < 0$, and $\sum_{k=2}^\infty \alpha_k < 0$. Hence we have

$$\int_2^\infty P_3(x)f(x)dx < \frac{1}{64}f(2). \tag{2.3}$$

In view of (2.2) and (2.3), we have (2.1).The lemma is prove.

Lemma 2.2. Define the function I(x) as

$$I(x) := \frac{2}{(1+x^2)^2} + \frac{1}{4+x^2} + \frac{11}{6(4+x^2)^2} + \frac{47}{12(4+x^2)^3} - \frac{5}{(4+x^2)^4} \ (x \in R). \tag{2.4}$$

Then, (i) if $|c| > 1$, we have

$$\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8} < \frac{1}{c^2+1/6}; \tag{2.5}$$

(ii) if $|c| \leq 1$, we have

$$I(c) < \frac{1}{c^2+1/6}. \tag{2.6}$$

Proof. (i) When $|c| > 1$, inequality (2.5) is equivalent to the following

$$1 - [\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} + \frac{57}{2048c^8}](c^2+1/6)$$
$$= (\frac{11c^4}{180} - \frac{2053c^2}{92160} - \frac{57}{12288})\frac{1}{c^8} > 0. \tag{2.5$'$}$$

Obviously (2.5)$'$ is always valid when $|c| > 1$. Hence (2.5) is true.

(ii)If $|c| \leq 1$, replacing $1/(4+c^2)$ by y in (2.6), we obtain after simplifications,

$$g(y) := 88 - 1393y + 8010y^2 - 21249y^3 + 12420y^4 < 0, y \in [1/5, 1/4]. \tag{2.6$'$}$$

In face, since $g^{(4)}(y) > 0, g^{(3)}(y)$ is increasing in $[1/5, 1/4]$, we can find that $max\{g^{(3)}(y); 1/5 \leq y \leq 1/4\} = g^{(3)}(1/4) < 0$, whence $g^{(3)}(y) < 0$. It follows that $g^{(2)}(y)$ is decreasing in $[1/5, 1/4]$. We find also that $max\{g^{(2)}(y); 1/5 \leq y \leq 1/4\} = g^{(2)}(1/5) < 0$, whence $g^{(2)}(y) < 0$. Hence $g'(y)$ is decreasing in $[1/5, 1/4]$. Since $g'(1/5) < 0$, then $g'(y) < 0$, and g(y) is increasing in [1/5,1/4]. At last we find that $g(1/5) < 0$, whence $g(y) < 0$. Consequently (2.6)$'$ is true; so is (2.6).The lemma is proved.

Lemma 2.3. Define the function J(x) as

$$J(x) := \frac{2}{(1+x^2)^2} + \frac{1}{4+x^2} + \frac{11}{6(4+x^2)^2} + \frac{65}{48(4+x^2)^3}$$
$$+\frac{8}{(4+x^2)^4} - \frac{16}{(4+x^2)^5} \ (x \in R). \tag{2.7}$$

Then, (i) if $c^2 > 5/3$, we have

$$\frac{1}{c^2+5/12} < \frac{1}{c^2} - \frac{1}{6c^4} - \frac{23}{256c^6}; \tag{2.8}$$

(ii) if $c^2 \leq 5/3$, we have

$$J(c) > \frac{1}{c^2+12/5}. \tag{2.9}$$

Proof. (i) When $c^2 > 5/3$, inequality (2.8) is equivalent to the following

$$c^4 - \frac{367c^2}{576} - \frac{115}{768} > 0. \tag{2.8$'$}$$

Obviously (2.8)$'$ is valid for all c ($c^2 > 5/3$). Hence (2.8) is true.

(ii) Replacing $1/(4+c^2)$ by y in (2.9), we obtain easily after simplifications,

$$h(y) := 3 - \frac{1084y}{48} + \frac{10765y^2}{48} - \frac{63642y^3}{48} + \frac{203709y^4}{48}$$

$$-8952y^5 + 6192y^6 > 0 \ (y \in [\frac{3}{17}, \frac{1}{4}]). \tag{2.9$'$}$$

In face, $h^{(4)}(y) = 203709/2 - 1074240y + 2229120y^2, h^{(4)}(3/17) = -18298.3027 < 0$, and $h^{(4)}(1/4) = -27385.5 < 0$, then $h^{(4)}(y) < 0$, and $h^{(3)}$ is decreasing in [3/17,1/4]. We can find that $max\{h^{(3)}(y); y \in [3/17, 1/4]\} = h^{(3)}(3/17) = -2624.37853 < 0$, then $h^{(3)}(y) < 0$, and $h^{(2)}(y)$ is decreasing in [3/17,1/4]. We find that $h^{(2)}(3/17) = -397.41057 < 0$, then $h^{(2)}(y) < 0$, and $h'(y)$ is decreasing in [3/17,1/4]. Since $h'(3/17) = -11.0578809 < 0$, then $h'(y) < 0$, and h(y) is increasing in [3/17,1/4]. At last, we find that $h(1/4) = 18.64689127 > 0$, whence $h(y) > 0$ and (2.9)$'$ is true; so is (2.9). The lemma is proved.

3. MAIN RESULT AND REMARK

Theorem 3.1. Let s(c) be Mathieu's series defined by (1.1). Then we have

$$\frac{1}{c^2 + 5/12} < s(c) < \frac{1}{c^2 + 1/6} \ (c \in R), \tag{3.1}$$

where the constant 1/6 is the best possible.

Proof. Let f(x) be a function which is continuous and qth differential. Using Euler-Maclaurin's formula(see [5]), we have

$$\sum_{k=n+1}^{m} f(k) = \int_n^m f(x)dx + \sum_{k=1}^{q} \frac{(-1)^k}{k!} B_k f^{(k-1)}(x)|_n^m$$

$$+\frac{(-1)^{q+1}}{q!} \int_n^m P_q(x) f^{(q)}(x)dx, \tag{3.2}$$

where $m > n, n, m \in N$. Let q=3, n=2, and $m \to \infty$. we can obtain

$$\sum_{k=2}^{\infty} f(k) = \int_2^{\infty} f(x)dx + \frac{1}{2}f(2) - \frac{1}{12}f'(2) + \frac{1}{6}\int_2^{\infty} P_3(x) f^{(3)}(x)dx. \tag{3.3}$$

Assume that $f(x) := 2x/\big(x^2+c^2\big)^2$. By computation, we find the following results:

$$\int_2^{\infty} f(x)dx = \frac{1}{4+c^2}; f(2) = \frac{4}{(4+c^2)^2}; f'(2) = \frac{2}{(4+c^2)^2} - \frac{32}{(4+c^2)^3};$$

$$f^{(3)}(x) = \frac{480c^2}{(x^2+c^2)^4} - [\frac{120}{(x^2+c^2)^3} + \frac{384c^4}{(x^2+c^2)^5}].$$

Hence by (3.3), we have

$$s(c) = \frac{2}{(1+c^2)^2} + \sum_{n=2}^{\infty} \frac{2n}{(n^2+c^2)^2} = \frac{2}{(1+c^2)^2} + \frac{1}{4+c^2} + \frac{11}{6(4+c^2)^2} + \frac{8}{3(4+c^2)^3}$$

$$-\frac{1}{6}\int_2^{\infty} P_3(x)[\frac{120}{(x^2+c^2)^3} + \frac{384c^4}{(x^2+c^2)^5}]dx + \frac{1}{6}\int_2^{\infty} P_3(x)\frac{480c^2}{(x^2+c^2)^4}dx. \tag{3.4}$$

By Lemma 2.1, we have

$$0 < \int_2^\infty P_3(x)\frac{480c^2}{(x^2+c^2)^4}dx < \frac{480c^2}{64(4+c^2)^4} = \frac{30}{4(4+c^2)^3} - \frac{30}{(4+c^2)^4};$$

$$-\frac{1}{64}\Big[\frac{120}{(4+c^2)^3} + \frac{384c^4}{(4+c^2)^5}\Big] < -\int_2^\infty P_3(x)\Big[\frac{120}{(x^2+c^2)^3} + \frac{384c^4}{(x^2+c^2)^5}\Big]dx < 0.$$

Consequently, (3.4) can be reduced to $J(c) < s(c) < I(c)$, where I(x) and J(x) are defined respectively by (2.4) and (2.7). In virtue of (2.8) and (2.9), and in view of the left-hand side of (1.4), it follows that the inequality of the left-hand side of (3.1) is valid. It is know from the right-hand side of (1.5),(2.5) and (2.6) that the right-hand side of (3.1) is also valid.

We now show that for any $\epsilon > 0$, there exists $c_0 > 0$, such for $c > c_0$, that

$$s(c) \geq \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1+\epsilon}{30c^6} + \frac{57}{2048c^8}. \tag{3.5}$$

Otherwise, by (1.5), we may get an increasing sequence $\{c_n; c_n \uparrow \infty\}$, which provides the following inequality

$$\frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1}{30c_n^6} - \frac{153}{2048c_n^8} < s(c_n) < \frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1+\epsilon}{30c_n^6} + \frac{57}{2048c_n^8};$$

$$\Big(\frac{\epsilon}{30}c_n^2 - \frac{210}{2048}\Big)\frac{1}{c_n^6} < s(c_n) - \Big(\frac{1}{c_n^2} - \frac{1}{6c_n^4} - \frac{1+\epsilon}{30c_n^6} + \frac{57}{2048c_n^8}\Big) < 0.$$

When n is large enough, we have $\big(\frac{\epsilon}{30}c_n^2 - \frac{210}{2048}\big)\frac{1}{c_n^6} > 0$. This is a contradictory. Then (3.5) is valid. Hence we may find a real number c, such c $(c > c_0)$ is larger enough, that

$$\begin{aligned} 1 - s(c)\big(c^2 + \frac{1}{6} + \epsilon\big) &\leq 1 - \Big[\frac{1}{c^2} - \frac{1}{6c^4} - \frac{1+\epsilon}{30c^6} + \frac{57}{2048c^8}\Big]\big(c^2 + \frac{1}{6} + \epsilon\big) \\ &= -\big(\epsilon - \frac{11+36\epsilon}{180c^2} - \cdots\big)\frac{1}{c^2}. \end{aligned} \tag{3.6}$$

In virtue of (3.6), we obtain that $s(c) > \frac{1}{c^2+\epsilon+1/6}$, for this c. Hence the constant 1/6 in (3.1) is the best possible. Thus we complete the proof of the theorem.

Remark 3.2. New inequality (3.1) is always valid for any real number c. Evidently the obtained result (3.1) is a bilateral improvement of (1.2). Moreover it is more succinct than (1.3). Particularly we point out that it is better than (1.3) when $|c| < 1$. At last we mention that it is impossible to improve the left-hand side of (3.1) any more, because of the constant 1/6 being the best possible.

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