INEQUALITIES OBTAINED FROM THE GENERALISATION OF OSTROWSKI'S INEQUALITY

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Abstract. In this article, from the generalization of Ostrowski's inequality we will obtain other inequalities.

1. Introduction

The following generalization of the Ostrowski inequality holds:

Theorem 1. *Let* $f:[a,b]\to\mathbb{R}$ *be absolutely continuous on* $[a,b]$ *and whose derivative* $f\prime:[a,b]\to\mathbb{R}$ *is bounded on* $[a,b]$. *Denote* $||f\prime||_\infty = ess\sup_{x\in[a,b]}|f'(x)|<\infty$. *Then*

$$(1.1)\qquad \left|\int_a^b f(x)dx-\left[(1-t)\,f(\alpha)+t\frac{f(a)+f(b)}{2}\right](b-a)\right| \leq \left[\frac{1}{4}(b-a)^2\left(2t^2-2t+1\right)+\left(\alpha-\frac{a+b}{2}\right)^2\right]||f'||_\infty$$

for all $t\in[0,1]$ *and all* α *such that* $a+\frac{b-a}{2}t\leq\alpha\leq b-\frac{b-a}{2}t$. *For the proof we refer the reader to* [1].

2. Preliminaries

Next we will give some auxiliary results.

Lemma 1. *Let* $f:[a,b]\to\mathbb{R}$ *be absolutely continuous on* $[a,b]$, *whose derivative* $f':[a,b]\to\mathbb{R}$ *is bounded on* $[a,b]$. *Then*

$$(2.1)\qquad 0\leq\frac{1}{2}(b-a)^2\,||f'||_\infty\,t^2 + (b-a)\left[-\frac{1}{2}(b-a)\,||f'||_\infty+\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)\right]t + \frac{1}{4}(b-a)^2\,||f'||_\infty+\left(\alpha-\frac{a+b}{2}\right)^2||f'||_\infty+\left(\int_a^b f(x)\,dx-f(\alpha)(b-a)\right)$$

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and

$$0 \leq \frac{1}{2}(b-a)^2 ||f'||_\infty t^2 + (b-a)\left[-\frac{1}{2}(b-a)||f'||_\infty - \left(f(\alpha) - \frac{f(a)+f(b)}{2}\right)\right] t + \frac{1}{4}(b-a)^2 ||f'||_\infty + \left(\alpha - \frac{a+b}{2}\right)^2 ||f'||_\infty - \left(\int_a^b f(x)\,dx - f(\alpha)(b-a)\right) \tag{2.2}$$

for all $t \in [0,1]$ and all α such that $a + \frac{b-a}{2}t \leq \alpha \leq b - \frac{b-a}{2}t$.

Proof. From Theorem 1 we have that

$$\begin{aligned} &-\left[\frac{1}{4}(b-a)^2\left(2t^2-2t+1\right) + \left(\alpha - \frac{a+b}{2}\right)^2\right] ||f'||_\infty \\ &\leq \int_a^b f(x)\,dx - \left[(1-t)f(\alpha) + t\frac{f(a)+f(b)}{2}\right](b-a) \\ &\leq \left[\frac{1}{4}(b-a)^2\left(2t^2-2t+1\right) + \left(\alpha - \frac{a+b}{2}\right)^2\right] ||f'||_\infty \end{aligned}$$

for all $t \in [0,1]$ and all α such that

$$a + \frac{b-a}{2}t \leq \alpha \leq b - \frac{b-a}{2}t,$$

from where (2.1) and (2.2) result. □

Next, let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$, whose derivative $f' : [a,b] \to \mathbb{R}$ is bounded on $[a,b]$ and $f' \neq 0$.

Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be two second degree functions, given by

$$f_1(t) = \frac{1}{2}(b-a)^2 ||f'||_\infty t^2 + (b-a)\left[-\frac{1}{2}(b-a)||f'||_\infty + \left(f(\alpha) - \frac{f(a)+f(b)}{2}\right)\right] t + \frac{1}{4}(b-a)^2 ||f'||_\infty + \left(\alpha - \frac{a+b}{2}\right)^2 ||f'||_\infty + \left(\int_a^b f(x)\,dx - f(\alpha)(b-a)\right) \tag{2.3}$$

and

$$f_2(t) = \frac{1}{2}(b-a)^2 ||f'||_\infty t^2 + (b-a)\left[-\frac{1}{2}(b-a)||f'||_\infty - \left(f(\alpha) - \frac{f(a)+f(b)}{2}\right)\right] t + \frac{1}{4}(b-a)^2 ||f'||_\infty + \left(\alpha - \frac{a+b}{2}\right)^2 ||f'||_\infty - \left(\int_a^b f(x)\,dx - f(\alpha)(b-a)\right) \tag{2.4}$$

for all $t \in \mathbb{R}$ and $V_1(t_1, y_1)$, $V_2(t_2, y_2)$ the apex of the parabolas generated by the functions f_1 and f_2.

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous mapping on $[a,b]$, differentiable mapping on (a,b), whose derivative $f' : [a,b] \to \mathbb{R}$ is bounded on $[a,b]$ and $f' \neq 0$. Then*

$$-\frac{1}{2} \leq \frac{f(\alpha) - \frac{f(a)+f(b)}{2}}{(b-a)\,||f'||_\infty} \leq \frac{1}{2}. \tag{2.5}$$

for all $t \in [0,1]$ and all α such that

$$a + \frac{b-a}{2}t \leq \alpha \leq b - \frac{b-a}{2}t$$

and

$$0 \leq t_k \leq 1 \tag{2.6}$$

for $k \in \{0,1\}$.

Proof. From Lagrange' s theorem we have that there exists $\zeta_1 \in (a,\alpha)$ and $\zeta_2 \in (\alpha,b)$ such that $f(\alpha) - f(a) = (\alpha - a) f'_{(\xi 1)}$ and $f(\alpha) - f(b) = (\alpha - b) f'_{(\xi 2)}$, from where

$$f(\alpha) - \frac{f(a)+f(b)}{2} = \frac{1}{2}\left[(\alpha - a) f'_{(\xi 1)} + (\alpha - b) f'_{(\xi 2)}\right]. \tag{2.7}$$

However,

$$\begin{aligned}\left|\frac{1}{2}\left[(\alpha - a) f'_{(\xi 1)} + (\alpha - b) f'_{(\xi 2)}\right]\right| &\leq \frac{1}{2}\left[(\alpha - a)\,|f'_{(\xi 1)}| + (b-\alpha)\,|f'_{(\xi 2)}|\right] \\ &\leq \frac{1}{2}\left[(\alpha - a) + (b - \alpha)\right]||f'||_\infty \\ &= \frac{b-a}{2}||f'||_\infty\end{aligned}$$

and taking (2.7) into account, we obtain

$$\left|f(\alpha) - \frac{f(a)+f(b)}{2}\right| \leq \frac{b-a}{2}||f'||_\infty. \tag{2.8}$$

From (2.8) we obtain the result (2.5).

From (2.3) we have that

$$t_1 = \frac{\frac{b-a}{2}\cdot||f'||_\infty - \left(f(\alpha) - \frac{f(a)+f(b)}{2}\right)}{(b-a)\cdot||f'||_\infty},$$

from where

$$t_1 = \frac{1}{2} - \frac{f(\alpha) - \frac{f(a)+f(b)}{2}}{(b-a)\cdot||f'||_\infty} \tag{2.9}$$

and similarly

$$t_2 = \frac{1}{2} + \frac{f(\alpha) - \frac{f(a)+f(b)}{2}}{(b-a)\cdot||f'||_\infty}. \tag{2.10}$$

From (2.5), (2.9) and (2.10) we obtain the inequalities from (2.6). □

3. Main Results

Theorem 2. *Let the mapping $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), whose derivative $f':[a,b]\to\mathbb{R}$ is bounded on $[a,b]$ and $f'\neq 0$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(\alpha)+\frac{f(a)+f(b)}{2}}{2}\right| \le \frac{1}{8}(b-a)\,||f'||_\infty+\frac{(\alpha-\frac{a+b}{2})^2\cdot||f'||_\infty}{b-a}-\frac{\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)^2}{2\cdot||f'||_\infty(b-a)} \tag{3.1}$$

for all $t\in[0,1]$ and all α such that $a+\frac{b-a}{2}t\le\alpha\le b-\frac{b-a}{2}t$.

Proof. From Lemma 1 we have that

$$0\le f_1(t) \tag{3.2}$$

$$0\le f_2(t) \tag{3.3}$$

for all $t\in[0,1]$.

Taking into account the results of Lemma 2, relations (3.2) and (3.3) hold iff

$$y_k\ge 0 \tag{3.4}$$

for $k\in\{0,1\}$.

The inequalities from (3.4) are equivalent to $\Delta_1\le 0$ and $\Delta_2\le 0$, where Δ_1,Δ_2 are the discriminants of the second degree functions f_1 and f_2 respectively, thus

$$(b-a)^2\left[-\frac{1}{2}(b-a)\,||f'||_\infty+\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)\right]^2 -2(b-a)^2||f'||_\infty\left[\frac{1}{4}(b-a)^2||f'||_\infty+\left(\alpha-\frac{a+b}{2}\right)^2||f'||_\infty +\left(\int_a^b f(x)\,dx-f(\alpha)(b-a)\right)\right]\le 0$$

and

$$(b-a)^2\left[-\frac{1}{2}(b-a)\,||f'||_\infty-\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)\right]^2 -2(b-a)^2||f'||_\infty\left[\frac{1}{4}(b-a)^2||f'||_\infty+\left(\alpha-\frac{a+b}{2}\right)^2||f'||_\infty -\left(\int_a^b f(x)\,dx-f(\alpha)(b-a)\right)\right]\le 0.$$

The inequalities from above give

$$-\left[\frac{1}{4}(b-a)^2\|f'\|_\infty^2+2\left(\alpha-\frac{a+b}{2}\right)^2\|f'\|_\infty^2-\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)^2\right] \tag{3.5}$$
$$\leq 2(b-a)\|f'\|_\infty\left[\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(\alpha)+\frac{f(a)+f(b)}{2}}{2}\right]$$
$$\leq \frac{1}{4}(b-a)^2\|f'\|_\infty+2\left(\alpha-\frac{a+b}{2}\right)^2\|f'\|_\infty^2-\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)^2$$

for all $t\in[0,1]$ and all α such that $a+\frac{b-a}{2}t\leq\alpha\leq b-\frac{b-a}{2}t$.

Taking into account (2.8), we have that

$$\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)^2\leq\frac{1}{4}(b-a)^2\|f'\|_\infty^2,$$

from where

$$\frac{1}{4}(b-a)^2\|f'\|_\infty^2-\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)^2\geq 0$$

and from (3.5) we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(\alpha)+\frac{f(a)+f(b)}{2}}{2}\right| \tag{3.6}$$
$$\leq\frac{\frac{1}{4}(b-a)^2\|f'\|_\infty^2+2\left(\alpha-\frac{a+b}{2}\right)^2\|f'\|_\infty^2-\left(f(\alpha)-\frac{f(a)+f(b)}{2}\right)^2}{2(b-a)\|f'\|_\infty}$$

for all $t\in[0,1]$ and all α such that $a+\frac{b-a}{2}t\leq\alpha\leq b-\frac{b-a}{2}t$.

From (3.6), inequality (3.1) results. □

Corollary 1. *Let the mapping $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), whose derivative $f':[a,b]\to\mathbb{R}$ is bounded on $[a,b]$ and $f'\neq 0$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(\alpha)+\frac{f(a)+f(b)}{2}}{2}\right| \tag{3.7}$$
$$\leq\frac{1}{8}(b-a)\|f'\|_\infty+\left(\alpha-\frac{a+b}{2}\right)^2\frac{\|f'\|_\infty}{b-a}$$

for all $t\in[0,1]$ and all α such that $a+\frac{b-a}{2}t\leq\alpha\leq b-\frac{b-a}{2}t$, we have the following mixture of the trapezoid and mid-point inequalities

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}}{2}\right| \tag{3.8}$$
$$\leq\frac{1}{8}(b-a)\|f'\|_\infty-\frac{\left(f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right)}{2(b-a)\|f'\|_\infty}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}}{2}\right|\leq\frac{1}{8}(b-a)\|f'\|_\infty. \tag{3.9}$$

Proof. The above corollary can be obtained by applying the results of Theorem 2. □

Remark 1. *The inequalities (3.7) and (3.9) can be found in* [1].

Theorem 3. *Let the mapping $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), whose derivative $f' : [a,b] \to \mathbb{R}$ is bounded on $[a,b]$ and $f' \neq 0$. If $I_n : a = x_0 < x_1 < .... < x_{n-1} < x_n = b$ is a partition of $[a,b]$ and $h_i = x_{i+1} - x_i$, $i \in \{0,1,...,n-1\}$, then*

$$\int_a^b f(x)\,dx = A(I_n,\xi,t,f) + R(I_n,\xi,t,f) \tag{3.10}$$

where

$$A(I_n,\xi,t,f) = \frac{1}{2}\sum_{i=0}^{n-1} f(\xi_i)\,h_i + \frac{1}{2}\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}h_i \tag{3.11}$$

for all $t \in [0,1]$ and all ξ_i such that $x_i + t\frac{h_i}{2} \le \xi_i \le x_{i+1} - t\frac{h_i}{2}$, $i \in 0\{0,1,...,n-1\}$ and the remainder $R(I_n,\xi,t,f)$ satisfies the estimation

$$|R(I_n,\xi,t,f)| \le \left[\frac{1}{8}\sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2\right]||f'||_\infty - \frac{1}{2||f'||_\infty}\sum_{i=0}^{n-1}\left(f(\xi_i) - \frac{f(x_i)+f(x_{i+1})}{2}\right)^2. \tag{3.12}$$

Proof. Applying Theorem 2 on the interval $[x_i, x_{i+1}]$, $i \in \{0,1,...,n-1\}$, we get

$$-\left\{\frac{1}{8}h_i^2 + \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2||f'||_\infty - \frac{1}{2||f'||_\infty}\left(f(\xi_i) - \frac{f(x_i)+f(x_{i+1})}{2}\right)^2\right\}$$

$$\le \int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{1}{2}f(\xi_i)\,h_i - \frac{1}{2}\frac{f(x_i)+f(x_{i+1})}{2}h_i$$

$$\le \left[\frac{1}{8}h_i^2 + \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2\right]||f'||_\infty - \frac{1}{2||f'||_\infty}\left(f(\xi_i) - \frac{f(x_i)+f(x_{i+1})}{2}\right)^2,$$

for all $t \in [0,1]$ and all ξ_i such that $x_i + t\frac{h_i}{2} \le \xi_i \le x_{i+1} - t\frac{h_i}{2}$, $i \in \{0,1,...,n-1\}$.

Summing over i from 0 to $n-1$ we get the estimation (3.12). □

Remark 2. *In Theorem 3, the member $\sum_{i=0}^{n-1} f(\xi_i)\,h_i$ is the Riemann' s sum and the number $\sum_{i=0}^{n-1}\frac{f(x_i)+(fx_{i+1})}{2}h_i$ is the trapezoidal rule. Then $A(I_n,\xi,t,f)$ is the arithmetic mean of the Riemann sum and the trapezoidal rule.*

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