A SIMPLE NEW PROOF OF FAN-TAUSSKY-TODD INEQUALITIES

ZHEN-GANG XIAO AND ZHI-HUA ZHANG

Abstract. In this paper we present simple new proofs of the inequalities:

$$2\left(1-\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n+1}(a_k-a_{k-1})^2 \le 2\left(1+\cos\frac{k\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2$$

which holds for all real numbers $a_0=0, a_1, \cdots, a_n, a_{n+1}=0$. The coefficients $2(1-\cos(\pi/(n+1)))$ and $2(1+\cos(\pi/(n+1)))$ are the best possible; and

$$2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n}(a_k-a_{k-1})^2 \le 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2$$

which holds for all real numbers $a_0=0, a_1, \cdots, a_n$ and the coefficients $2(1-\cos(\pi/(2n+1)))$ and $2(1+\cos(\pi/(2n+1)))$ are the best possible.

1. Introduction

In 1955, K. Fan, O. Taussky and J. Todd [2] published a remarkable paper proving discrete analogues of several well-known integral inequalities. Among their results is the following theorem:

Theorem 1.1. *Assume a_i are real numbers for $1 \le i \le n$, we have*

(a) if $a_0 = a_{n+1} = 0$, then

$$2\left(1-\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n+1}(a_k-a_{k-1})^2 \tag{1.1}$$

with equality holding if and only if $a_k = c\sin\frac{k\pi}{n+1}$ ($k=1,2,\cdots,n$, c is a real constant).

(b) if $a_0 = 0$, then

$$2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n}(a_k-a_{k-1})^2 \tag{1.2}$$

with the equality holding if and only if $a_k = c\sin\frac{k\pi}{2n+1}$ ($k=1,2,\cdots,n$, c is a real constant).

The two constants $2\left(1-\cos\frac{\pi}{n+1}\right)$ and $2\left(1-\cos\frac{\pi}{2n+1}\right)$ given in inequalities (1.1) and (1.2), respectively, are the best possible.

Redheffer [4] presented an "ingenious proof" for these results based on an analysis of the characteristic values and vectors of Hermitian matrices. The main tool is an intriguing inequality of D.E. Rutherford who investigated the structure of Hermitian matrices "because of their great importance in a number of mathematical models of chemical and physical processes" [1]. E.F. Beckenbach and R. Bellman mention the Theorem 1.1 as well as similar results are important for the numerical integration of differential equations. Motivated to find "easy proofs" of the inequalities (1.1) and (1.2), R.M. Redheffer [4] presented in 1983 a very elegant elementary method for the proof of Theorem 1.1.

1991 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Simple proof, Fan-Taussky-Todd inequalities, reverse.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

In 1982, G.V. Milovanovic and I.Z. Milovanovic [3] obtained the reverses of inequalities (1.1) and (1.2). By using techniques similar to those of Fan-Taussky-Todd, they proved:

Theorem 1.2. *Assume a_i are real numbers for $1 \le i \le n$, we have*
(a) if $a_0 = a_{n+1} = 0$, then

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{k\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \tag{1.3}$$

with the equality holding if and only if $a_k = (-1)^{k-1} c \sin\frac{k\pi}{n+1}(k=1,2,\cdots,n,$ c is a real constant).
(b) if $a_0 = 0$, then

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \tag{1.4}$$

with the equality holding if and only if $a_k = (\ -1)^{k-1} c \sin\frac{k\pi}{2n+1}(k=1,2,\cdots,n,$ c is a real constant).

By using a modification of Redheffer's technique [4], H.Alzer [5] gave a simpler proof of inequalities (1.3) and (1.4) in 1991.

Combining Theorem 1.1 and Theorem 1.2, we obtain

Theorem 1.3. *Assume a_i are real numbers for $1 \le i \le n$, we have*
(a) if $a_0 = a_{n+1} = 0$, then

$$2\left(1-\cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{k\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \tag{1.5}$$

and the coefficients $2(1-\cos\frac{\pi}{n+1})$ and $2(1+\cos\frac{\pi}{n+1})$ are the best possible.
(b) if $a_0 = 0$, then

$$2\left(1-\cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \le \sum_{k=1}^{n} (a_k - a_{k-1})^2 \le 2\left(1+\cos\frac{2\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \tag{1.6}$$

and the coefficients $2(1-\cos\frac{\pi}{2n+1})$ and $2(1+\cos\frac{\pi}{2n+1})$ are the best possible.

Therefore (1.1) and (1.2) are called Fan-Taussky-Todd inequalities, and (1.3) and (1.4) are called the reverse Fan-Taussky-Todd inequalities.

In this paper, we give a simple elementary proof of inequalities (1.5) and (1.6).

2. Main Result

Theorem 2.1. *If n is an integer for $n > 1, a_i (1 \le i \le n)$ be n real numbers, and $t \in \left(0, \frac{\pi}{n}\right)$, then we have*

$$2\cos t \sum_{k=1}^{n} a_k^2 \ge \frac{\sin(n+1)t}{\sin nt} a_n^2 \pm 2\sum_{k=1}^{n-1} a_k a_{k+1} \tag{2.1}$$

with the equality holding if and only if $a_k = c_1 \sin kt$ for "+", and $a_k = (-1)^{k-1} c_2 \sin kt$ for "−", where $k = 1, 2, \cdots, n$, and c_1, c_2 be two real constants.

Proof. From $t \in \left(0, \frac{\pi}{n}\right)$, we have $\sin kt > 0 (k = 1, 2, \cdots, n-1)$, and

$$\frac{\sin(k+1)t}{\sin kt} a_k^2 + \frac{\sin kt}{\sin(k+1)t} a_{k+1}^2 \ge \pm 2 a_k a_{k+1} \tag{2.2}$$

with equality holding if and only if

$$a_{k+1} \sin kt \pm a_k \sin(k+1)t = 0.$$

From (2.2), summing from 1 to $n-1$, we get

$$\sum_{k=1}^{n-1}\left[\frac{\sin(k+1)t}{\sin kt}a_k^2+\frac{\sin kt}{\sin(k+1)t}a_{k+1}^2\right]\geq \pm 2\sum_{k=1}^{n-1}a_ka_{k+1}.$$

that is

$$\sum_{k=1}^{n}\frac{\sin(k-1)t+\sin(k+1)t}{\sin kt}a_k^2\geq \frac{\sin(n+1)t}{\sin nt}a_k^2\pm 2\sum_{k=1}^{n-1}a_ka_{k+1}. \tag{2.3}$$

Utilizing the fact that

$$\sin(k-1)t+\sin(k+1)t=2\sin kt\cos t,$$

inequalities (2.3) becomes (2.1), with equality holding if and only if

$$a_{k+1}\sin kt=a_k\sin(k+1)t$$

or $a_k=c\sin kt$ ($k=1,2,\cdots,n,c$ is a constant) for "+"; and

$$a_{k+1}\sin kt+a_k\sin(k+1)t=0$$

or $a_k=(-1)^{k-1}c\sin kt$ ($k=1,2,\cdots,n$, c is a constant) for "$-$". This proves Theorem 2.1. □

3. Remarks

Remark 3.1. *Let (2.1) for "+", let $t=\frac{\pi}{n+1}$, or $t=\frac{\pi}{2n+1}$, since*

$$\sin\pi=0,\quad \sin\frac{(n+1)\pi}{2n+1}=\sin\frac{\pi}{2n+1},$$

then we have

$$\cos\frac{\pi}{n+1}\sum_{k=1}^{n}a_k^2\geq\sum_{k=1}^{n}a_ka_{k+1} \tag{3.1}$$

with the equality holding if and only if $a_k=c\sin\frac{k\pi}{n+1}$($k=1,2,\cdots,n$, c is a constant), and

$$2\cos\frac{\pi}{2n+1}\sum_{k=1}^{n}a_k^2\geq a_n^2+2\sum_{k=1}^{n}a_ka_{k+1} \tag{3.2}$$

with the equality holding if and only if $a_k=c\sin\frac{k\pi}{2n+1}$($k=1,2,\cdots,n$, c is a constant), respectively.

Remark 3.2. *Let (2.1) for "$-$", and $t=\frac{\pi}{n+1},\frac{2\pi}{n+1}$, because*

$$\sin\frac{2(n+1)\pi}{2n+1}=\sin\frac{2n\pi}{2n+1},$$

therefore

$$\cos\frac{\pi}{n+1}\sum_{k=1}^{n}a_k^2\geq-\sum_{k=1}^{n-1}a_ka_{k+1} \tag{3.3}$$

with the equality holding if and only if $a_k=(-1)^{k-1}c\sin\frac{k\pi}{n+1}$($k=1,2,\cdots,n$, c is a constant), and

$$2\cos\frac{2\pi}{2n+1}\sum_{k=1}^{n}a_k^2\geq -a_n^2-2\sum_{k=1}^{n-1}a_ka_{k+1} \tag{3.4}$$

with the equality holding if and only if $a_k=(-1)^{k-1}c\sin\frac{k\pi}{2n+1}$($k=1,2,\cdots,n$, c is a constant), respectively.

Remark 3.3. *It is easy to see that the equalities (3.1) and (3.3) are equivalent to (1.5), and the equalities (3.2) and (3.4) are equivalent to (1.6), respectively.*

Remark 3.4. *Combining inequalities (3.1) and (3.3), that is to say: assume* $a_i (1 \leq i \leq n)$ *be* n *real numbers for* $\sum_{i=1}^{n} a_i^2 \neq 0$*, and*

$$f(a_1, a_2, \cdots, a_n) = \frac{\sum\limits_{k=1}^{n-1} a_k a_{k+1}}{\sum\limits_{k=1}^{n} a_k^2} \tag{3.5}$$

then

$$f_{max} = \cos \frac{\pi}{n+1} \tag{3.6}$$

and

$$f_{min} = -\cos \frac{\pi}{n+1}. \tag{3.7}$$

References

[1] E.F.Beckenbach and R.Bellman. *"Inequalities"*, Springer, Berlin, 1983.
[2] Fan.k,O.Taussky and J.Todd.*Discrete analogues of inequalities of wirtinger.* Monatsh,Math, 59(1995),73-90.
[3] G.V.Milovanovic and I.Z.Milovanovic. *On discrete inequalities of Wirtinger's type.* J.Math Anal,Appl, 88(1982)378-387.
[4] R.M.Redheffer. *Easy proof of hard inequalities*, in "General Inequalities 3"(E.F.Beckenbach,Ed.)123-140, Birkhauset, basel, 1983.
[5] H.Alzer. *Converses of two inequalities of Ky Fan, O.Taussky and J.Todd.* JMAA, 161(1991) 142-147.

(Zh.-G. Xiao) Department of Mathematics,Hunan Institute of science and Technology, Yueyang, Hunan 423400,P.R.China.
E-mail address: xiaozg@163.com

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400,P.R.China.
E-mail address: zxzh1234@163.com