

# PRE-GRÜSS TYPE INEQUALITIES IN INNER PRODUCT SPACES

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ABSTRACT. Some pre-Grüss type inequalities in real or complex inner product spaces and applications for integrals are given.

## 1. INTRODUCTION

Let  $f, g$  be two functions defined and integrable on  $[a, b]$ . Assume that

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma$$

for each  $x \in [a, b]$ , where  $\varphi, \Phi, \gamma, \Gamma$  are given real constants. Then the following inequality is well known in the literature as the Grüss inequality ([4, pp. 296])

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

In this inequality, G. Grüss has proven that, the constant  $\frac{1}{4}$  is the best possible in the sense that it cannot be replaced by a smaller one, and is achieved for

$$f(x) = g(x) = \operatorname{sgn} \left( x - \frac{a+b}{2} \right).$$

Recently, S. S. Dragomir has proved the following Grüss' type inequality in real or complex inner product spaces [1].

**Theorem 1.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ , ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0,$$

*hold, then we have the inequality*

$$(1.1) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

*The constant  $\frac{1}{4}$  is best possible in sense that it cannot be replaced by a smaller constant.*

In [2], by using the following lemmas

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**Lemma 1.** *Let  $x, e \in H$  with  $\|e\| = 1$  and  $\delta, \Delta \in \mathbb{K}$  with  $\delta \neq \Delta$ . Then*

$$\operatorname{Re} \langle \Delta e - x, x - \delta e \rangle \geq 0$$

*if and only if*

$$\left\| x - \frac{\delta + \Delta}{2} e \right\| \leq \frac{1}{2} |\Delta - \delta|,$$

and

**Lemma 2.** *Let  $x, e \in H$  with  $\|e\| = 1$ . Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2,$$

the author gave an alternative proof for (1.1) and also obtained the following refinement of it, namely

**Theorem 2.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \gamma, \Phi, \Gamma$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that either the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0, \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

*hold, then we have the inequality*

$$\begin{aligned} & |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \\ & \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \\ & \leq \left( \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \right). \end{aligned}$$

*The constant  $\frac{1}{4}$  is best possible.*

Further, as a generalization for orthonormal families of vectors in inner product spaces, S.S. Dragomir proved, in [3], the following reverse of Bessel's inequality:

**Theorem 3.** *Let  $\{e_i\}$ ,  $i \in I$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\varphi_i, \Phi_i \in \mathbb{K}$ ,  $i \in F$  and  $x$  is vector in  $H$  such that either the condition*

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0,$$

*or equivalently,*

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}},$$

*holds, then we have the following reverse of Bessel's inequality*

$$(1.2) \quad \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \varphi_i|^2 - \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2.$$

*The constant  $\frac{1}{4}$  is best possible.*

The corresponding Grüss' type inequality is embodied in the following theorem:

**Theorem 4.** *Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\phi_i, \gamma_i, \Phi_i, \Gamma_i \in \mathbb{R}$  ( $i \in F$ ), and  $x, y \in H$ . If either*

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{i=1}^n \Phi_i e_i - x, x - \sum_{i=1}^n \phi_i e_i \right\rangle &\geq 0, \\ \operatorname{Re} \left\langle \sum_{i=1}^n \Gamma_i e_i - y, y - \sum_{i=1}^n \gamma_i e_i \right\rangle &\geq 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \\ \left\| y - \sum_{i \in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| &\leq \frac{1}{2} \left( \sum_{i \in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

hold true, then

$$\begin{aligned} 0 &\leq \left| \langle x, y \rangle - \sum_{i=1}^n \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ &\leq \frac{1}{4} \left( \sum_{i=1}^n |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^n |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &\quad - \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \langle x, e_i \rangle \right| \left| \frac{\Gamma_i + \gamma_i}{2} - \langle y, e_i \rangle \right| \\ &\left( \leq \frac{1}{4} \left( \sum_{i=1}^n |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^n |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

The constant  $\frac{1}{4}$  is best possible in the sense that it cannot be replaced by a smaller constant.

The main aim of this paper is to provide some similar inequalities which, providing refinements of the usual Grüss' inequality, are known in the literature as pre-Grüss type inequalities. Applications for Lebesgue integrals in general measure spaces are also given.

## 2. PRE-GRÜSS INEQUALITIES IN INNER PRODUCT SPACES

We start with the following result:

**Theorem 5.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over  $\mathbb{K}$ , ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \Phi$  are real or complex numbers and  $x, y$  are vectors in  $H$  such that either the condition*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0,$$

or equivalently,

$$(2.1) \quad \left\| x - \frac{\varphi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \varphi|,$$

holds true, then we have the inequalities

$$(2.2) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} |\Phi - \varphi| \cdot \sqrt{(\|y\|^2 - |\langle y, e \rangle|^2)}$$

and

$$(2.3) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} |\Phi - \varphi| \cdot \|y\| - (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \cdot |\langle y, e \rangle|.$$

*Proof.* It is obvious that:

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle.$$

Using Schwarz's inequality in inner product spaces  $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$  for the vectors  $x - \langle x, e \rangle e$  and  $y - \langle y, e \rangle e$ , we deduce:

$$(2.4) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq (\|x\|^2 - |\langle x, e \rangle|^2) \cdot (\|y\|^2 - |\langle y, e \rangle|^2).$$

Now, the inequality (2.2) is a simple consequence of (1.1) for  $x = y$ , or of Lemma 2 and (2.1).

Since (see for instance [1]),

$$(2.5) \quad \begin{aligned} & \|x\|^2 - |\langle x, e \rangle|^2 \\ &= \operatorname{Re}((\Phi - \langle x, e \rangle) \cdot (\langle e, x \rangle - \bar{\varphi})) - \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle, \end{aligned}$$

then making use of the elementary inequality  $4 \operatorname{Re}(a\bar{b}) \leq |a + b|^2$  with  $a, b \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ), we can state that

$$(2.6) \quad \operatorname{Re}((\Phi - \langle x, e \rangle) \cdot (\langle e, x \rangle - \bar{\varphi})) \leq \frac{1}{4} |\Phi - \varphi|^2.$$

Using (2.5) and (2.6) we have

$$(2.7) \quad \|x\|^2 - |\langle x, e \rangle|^2 \leq \left( \frac{1}{2} |\Phi - \varphi| \right)^2 - \left( (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \right)^2.$$

Taking into account the inequalities (2.4) and (2.7), we get that

$$\begin{aligned} & |\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle|^2 \\ & \leq \left( \left( \frac{1}{2} |\Phi - \varphi| \right)^2 - \left( (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \right)^2 \right) \cdot (\|y\|^2 - |\langle y, e \rangle|^2). \end{aligned}$$

Finally, using the elementary inequality for positive real numbers:

$$(2.8) \quad (m^2 - n^2) \cdot (p^2 - q^2) \leq (mp - nq)^2$$

we have:

$$\begin{aligned} & \left( \left( \frac{1}{2} |\Phi - \varphi| \right)^2 - \left( (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \right)^2 \right) \cdot (\|y\|^2 - |\langle y, e \rangle|^2) \\ & \leq \left( \frac{1}{2} |\Phi - \varphi| \cdot \|y\| - (\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle)^{\frac{1}{2}} \cdot |\langle y, e \rangle| \right)^2, \end{aligned}$$

giving the desired inequality (2.3).  $\square$

A similar version for Bessel's inequality is incorporated in the following theorem:

**Theorem 6.** Let  $\{e_i\}_{i \in I}$ , be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\varphi_i, \Phi_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y$  are vectors in  $H$  such that either the condition

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0,$$

or equivalently,

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}}$$

holds. Then we have inequalities

$$(2.9) \quad \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \sqrt{\left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)}$$

and

$$(2.10) \quad \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \cdot \|y\| - \left( \sum_{i \in F} \left| \frac{\Phi_i + \varphi_i}{2} - \langle x, e_i \rangle \right|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)^{\frac{1}{2}}.$$

*Proof.* It is obvious (see for example [3]) that :

$$\langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle = \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle.$$

Using Schwarz's inequality in inner product spaces, we have:

$$(2.11) \quad \left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \leq \left\| x - \sum_{i \in F} \langle x, e_i \rangle e_i \right\|^2 \cdot \left\| y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\|^2 = \left( \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \cdot \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right).$$

In a similar manner to the one in the proof of Theorem 5 we may conclude that (2.9) holds true.

Now, using (1.2) and (2.11) we also have:

$$\begin{aligned} & \left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ & \leq \left( \frac{1}{2} \left( \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \right)^2 - \left( \left( \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \right)^{\frac{1}{2}} \right)^2 \right) \\ & \quad \times \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \end{aligned}$$

Finally, utilizing the elementary inequality (2.8), we have

$$\begin{aligned} (2.12) \quad & \left( \frac{1}{2} \left( \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \right)^2 - \left( \left( \sum_{i \in F} \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right)^2 \right)^{\frac{1}{2}} \right)^2 \\ & \times \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \\ & \leq \left( \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \cdot \|y\|^2 - \left( \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \right)^{\frac{1}{2}} \cdot \sum_{i \in F} |\langle y, e_i \rangle|^2 \right)^2 \end{aligned}$$

which gives the desired result (2.10).  $\square$

Another pre-Grüss type inequality associated to orthonormal families in inner product spaces is incorporated in the next theorem.

**Theorem 7.** *Let  $\{e_i\}$ ,  $i \in I$  be a family of orthonormal vectors in  $H$ ,  $F$  a finite part of  $I$ ,  $\varphi_i, \Phi_i \in \mathbb{K}$ ,  $i \in F$  and  $x, y$  vectors in  $H$  such that either the condition*

$$\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \varphi_i e_i \right\rangle \geq 0,$$

or equivalently,

$$\left\| x - \sum_{i \in F} \frac{\Phi_i + \varphi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}},$$

holds. Then we have inequalities

$$\begin{aligned} & \left| \langle x, y \rangle - \sum_{i \in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ & \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \cdot \|y\| - \sum_{i \in F} \left| \frac{\Phi_i + \varphi_i}{2} - \langle x, e_i \rangle \right| |\langle y, e_i \rangle|. \end{aligned}$$

*Proof.* Using Schwarz's inequality (2.11) with the reverse of Bessel's inequality (1.2) we have:

$$\begin{aligned} & \left| \left\langle x - \sum_{i \in F} \langle x, e_i \rangle e_i, y - \sum_{i \in F} \langle y, e_i \rangle e_i \right\rangle \right|^2 \\ & \leq \left( \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \right) \cdot \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \\ & \leq \left( \frac{1}{4} \sum_{i \in F} |\Phi_i - \varphi_i|^2 - \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \right) \cdot \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right). \end{aligned}$$

Further, on utilizing Aczél's inequality [4, p. 117] for two sequences of real numbers  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$  such that  $a_1^2 - a_2^2 - \dots - a_n^2 > 0$  or  $b_1^2 - b_2^2 - \dots - b_n^2 > 0$ , that is

$$(a_1^2 - a_2^2 - \dots - a_n^2) (b_1^2 - b_2^2 - \dots - b_n^2) \leq (a_1 b_1 - a_2 b_2 - \dots - a_n b_n)^2,$$

we have

$$\begin{aligned} & \left( \frac{1}{4} \sum_{i \in F} |\Phi_i - \varphi_i|^2 - \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right|^2 \right) \cdot \left( \|y\|^2 - \sum_{i \in F} |\langle y, e_i \rangle|^2 \right) \\ & \leq \left( \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \varphi_i|^2 \right)^{\frac{1}{2}} \|y\| - \sum_{i \in F} \left| \frac{\varphi_i + \Phi_i}{2} - \langle x, e_i \rangle \right| |\langle y, e_i \rangle| \right)^2. \end{aligned}$$

This completes the proof.  $\square$

### 3. APPLICATIONS FOR INTEGRALS

Let  $(\Omega, \Sigma, \mu)$  be a measure space consisting of a set  $\Omega$ ,  $\Sigma$  a  $\sigma$ -algebra of parts and  $\mu$  a countably additive and positive measure on  $\Sigma$  with values in  $\mathbb{R} \cup \{\infty\}$ . Denote by  $L^2(\Omega, \mathbb{K})$  the Hilbert space of all real or complex valued functions  $f$  defined on  $\Omega$  and 2-integrable on  $\Omega$ , i. e.

$$\int_{\Omega} |f(s)|^2 d\mu(s) < \infty.$$

The following proposition holds.

**Proposition 1.** *If  $f, g, h \in L^2(\Omega, \mathbb{K})$  and  $\varphi, \Phi \in \mathbb{K}$ , are such that  $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$  and, either*

$$(3.1) \quad \int_{\Omega} \operatorname{Re}((\Phi h(s) - f(s)) (\bar{f}(s) - \bar{\varphi} \bar{h}(s))) d\mu(s) \geq 0,$$

or, equivalently,

$$\left( \int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|,$$

holds, then we have the inequalities

$$\begin{aligned} & \left| \int_{\Omega} f(s) \bar{g}(s) d\mu(s) - \int_{\Omega} f(s) \bar{h}(s) d\mu(s) \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right| \\ & \leq \frac{1}{2} |\Phi - \varphi| \cdot \sqrt{\left( \int_{\Omega} |g(s)|^2 d\mu(s) - \left| \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right|^2 \right)} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} f(s) \bar{g}(s) d\mu(s) - \int_{\Omega} f(s) \bar{h}(s) d\mu(s) \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right| \\ & \leq \frac{1}{2} |\Phi - \varphi| \cdot \left( \int_{\Omega} |g(s)|^2 d\mu(s) \right)^{1/2} \\ & \quad - \left( \int_{\Omega} \operatorname{Re}((\Phi h(s) - f(s))(h(s) \bar{f}(s) - \varphi h(s))) d\mu(s) \right)^{\frac{1}{2}} \\ & \quad \times \left| \int_{\Omega} h(s) \bar{g}(s) d\mu(s) \right|. \end{aligned}$$

*Proof.* The proof follows by Theorem 5 on choosing  $H = L^2(\Omega, K)$  with the inner product

$$\langle f, g \rangle = \int_{\Omega} f(s) \bar{g}(s) d\mu(s).$$

□

**Remark 1.** We observe that, a sufficient condition for the condition (3.1) to hold, is that

$$(3.2) \quad \operatorname{Re}(\Phi h(s) - f(s)) (\bar{f}(s) - \overline{\varphi h}(s)) \geq 0,$$

for  $\mu$ -a.e.  $s \in \Omega$ .

If the functions are real-valued, then, for  $\Phi$  and  $\varphi$  real numbers, a sufficient condition for (3.2) to hold is

$$\Phi h(s) \geq f(s) \geq \varphi h(s)$$

for  $\mu$ -a.e.  $s \in \Omega$ .

In this way we can see the close connection that exists between the classical Grüss' inequality and the results we obtained above.

Now, consider the family  $\{f_i\}_{i \in I}$  of functions in  $L^2(\Omega, \mathbb{K})$  with the properties that

$$\int_{\Omega} f_i(s) \bar{f}_j(s) d\mu(s) = \delta_{ij}, \quad i, j \in I,$$

where  $\delta_{ij}$  is 0 if  $i \neq j$  and  $\delta_{ij} = 1$  if  $i = j$ .  $\{f_i\}_{i \in I}$  is an orthonormal family in  $L^2(\Omega, \mathbb{K})$ .

The following proposition holds.

**Proposition 2.** Let  $\{f_i\}_{i \in I}$  be an orthonormal family of functions in  $L^2(\Omega, \mathbb{K})$ ,  $F$  a finite subset of  $I$ ,  $\phi_i, \Phi_i \in \mathbb{K}$  ( $i \in F$ ) and  $f \in L^2(\Omega, \mathbb{K})$ , so that either

$$(3.3) \quad \int_{\Omega} \operatorname{Re} \left[ \left( \sum_{i \in F} \Phi_i f_i(s) - f(s) \right) \left( \bar{f}(s) - \sum_{i \in F} \overline{\phi_i} \bar{f}_i(s) \right) \right] d\mu(s) \geq 0$$



or, equivalently,

$$\int_{\Omega} \left| f(s) - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} f_i(s) \right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2.$$

holds. Then we have the inequalities

$$\begin{aligned} & \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \sum_{i \in F} \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \int_{\Omega} f_i(s) \overline{g(s)} d\mu(s) \right| \\ & \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \left( \int_{\Omega} |g(s)|^2 d\mu(s) - \sum_{i \in F} \left| \int_{\Omega} g(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \sum_{i \in F} \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \int_{\Omega} f_i(s) \overline{g(s)} d\mu(s) \right| \\ & \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{1/2} \left( \int_{\Omega} |g(s)|^2 d\mu(s) \right)^{1/2} \\ & \quad - \left( \sum_{i \in F} \left| \frac{\Phi_i + \phi_i}{2} - \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{1/2} \left( \sum_{i \in F} \left| \int_{\Omega} f(s) \overline{f_i(s)} d\mu(s) \right|^2 \right)^{1/2}. \end{aligned}$$

The proof is obvious by Theorem 7 and we omit the details.

**Remark 2.** In the real case, we observe that a sufficient condition for (3.3) to hold, is that

$$\sum_{i \in F} \Phi_i f_i(s) \geq f(s) \geq \sum_{i \in F} \varphi_i f_i(s)$$

for  $\mu$ -a.e.  $s \in \Omega$ .

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