SELF SHARPENING INEQUALITY

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Abstract. In this note we analyze the following inequality that is observed in the study of the density of n^{th}-power free integers

$$\left|\frac{x}{\zeta(n)}-f_n(x)\right|<\frac{\sqrt[n]{x}}{n-1}+\left(\frac{\sqrt[n]{x}}{\zeta(2)}-1\right)+\left(1+\frac{1}{\zeta(2)}\right)\sum_{i=1}^{M}x^{\frac{1}{n2^i}}+2x^{\frac{1}{n2^{M+1}}}\qquad(M\in\mathbb{N}),$$

in which $f_n(x)$ is the number of n^{th}-power free positive integers $\leq x$. We show that the above inequality is *self sharpening* and then we compute the order of its sharpening.

1. Introduction and Motivation

Let $\mathbb{P}$ be the set of all primes and suppose N is a positive integer, with the following prime factoring:

$$N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}\qquad(p_1,p_2,\cdots,p_k\in\mathbb{P}).$$

We say that N, is *n^{th}-power free* if all α_i's are less than n. Let $f_n(x)$ be the number of n^{th}-power frees $\leq x$. It is well-known that the density of n^{th}-power free integers is $\frac{1}{\zeta(n)}$, or equivalently

$$f_n(x)\sim\frac{x}{\zeta(n)}\qquad(x\to\infty),$$

such that $\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}$. In fact, we have

$$f_n(x)=\frac{x}{\zeta(n)}+O(\sqrt[n]{x}),$$

and the stronger result

$$\left|\frac{x}{\zeta(n)}-f_n(x)\right|<\frac{n}{n-1}\sqrt[n]{x}-1. \tag{1.1}$$

In particular,

$$f_2(x)<\frac{x}{\zeta(2)}+\sqrt{x}-1.$$

As you will see soon, during the proof of (1.1) we use $\left|\frac{x}{\zeta(n)}-f_n(x)\right|<\sum_{1<k\leq\sqrt[n]{x}}|\mu(k)|+\frac{\sqrt[n]{x}}{n-1}$ and $\sum_{1<k\leq\sqrt[n]{x}}|\mu(k)|<\sqrt[n]{x}-1$, in which μ is the well-known *Mobius Function* and defined by

$$\mu(m)=\begin{cases}1 & m=1;\\ (-1)^k & m=p_1p_2\cdots p_k;\\ 0 & \text{otherwise.}\end{cases}$$

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Since the function $\mu(k)$ vanishes if k is not square free, we can claim that

$$\sum_{1<k\leq \sqrt[n]{x}} |\mu(k)| \leq f_2(\sqrt[n]{x}) < \frac{\sqrt[n]{x}}{\zeta(2)} + \sqrt[2n]{x} - 1.$$

The obtained bound for $\sum_{1<k\leq \sqrt[n]{x}} |\mu(k)|$ is sharper than $\sqrt[n]{x} - 1$. Thus, (1.1) becomes sharper as follows:

$$\left|\frac{x}{\zeta(n)} - f_n(x)\right| < \frac{\sqrt[n]{x}}{n-1} + \frac{\sqrt[n]{x}}{\zeta(2)} + 2\sqrt[2n]{x} - 1. \tag{1.2}$$

By using (1.2) we have

$$f_2(x) < \frac{x}{\zeta(2)} + \left(1 + \frac{1}{\zeta(2)}\right)\sqrt{x} + 2\sqrt[4]{x} - 1,$$

and repeating above method we obtain,

$$\left|\frac{x}{\zeta(n)} - f_n(x)\right| < \frac{\sqrt[n]{x}}{n-1} + \frac{\sqrt[n]{x}}{\zeta(2)} + \left(1 + \frac{1}{\zeta(2)}\right)\sqrt[2n]{x} + 2\sqrt[4n]{x} - 1.$$

This process leads us to the following definition:

Definition 1 (Self Sharpening Inequality)**.** *We say that an inequality is self sharpening, if it sharp itself!*

In the next section we continue above method to show that (1.1) is self sharpening.

2. Proofs and Self Sharpening Inequality

For a proof of (1.1), we note that

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} = \frac{1}{\zeta(s)}.$$

Now, by a counting we have

$$f_n(x) = x - \sum_{p\in\mathbb{P}} \left\lfloor \frac{x}{p^n} \right\rfloor + \sum_{p,q\in\mathbb{P},p\neq q} \left\lfloor \frac{x}{(pq)^n} \right\rfloor - \cdots = \sum_{k\leq \sqrt[n]{x}} \mu(k) \left\lfloor \frac{x}{k^n} \right\rfloor.$$

and so,

$$\begin{aligned}\left|\frac{x}{\zeta(n)} - f_n(x)\right| &= \left|\sum_{1<k\leq \sqrt[n]{x}} \mu(k)\left(\frac{x}{k^n} - \left\lfloor \frac{x}{k^n}\right\rfloor\right) + \sum_{k>\sqrt[n]{x}} \mu(k)\frac{x}{k^n}\right| \\ &< \sum_{1<k\leq \sqrt[n]{x}} |\mu(k)| + x\sum_{k>\sqrt[n]{x}} \frac{1}{k^n} \\ &< \sqrt[n]{x} - 1 + x\int_{\sqrt[n]{x}}^{\infty} \frac{ds}{s^n} = \frac{n}{n-1}\sqrt[n]{x} - 1.\end{aligned}$$

Now, we show that (1.1) is self sharpening.

Theorem 1. *For all $M \in \mathbb{N}$ the inequality*

$$\left|\frac{x}{\zeta(n)} - f_n(x)\right| < \frac{\sqrt[n]{x}}{n-1} + \left(\frac{\sqrt[n]{x}}{\zeta(2)} - 1\right) + \left(1 + \frac{1}{\zeta(2)}\right)\sum_{i=1}^{M} x^{\frac{1}{n2^i}} + 2x^{\frac{1}{n2^{M+1}}}, \tag{2.1}$$

is self sharpening and the order of its sharpening is

$$O\left(x^{\frac{1}{n2^{M+1}}}\right).$$

Proof. The inequality (2.1) yields by induction on M and using (1.1). Now, let $U(M)$ be the right side of (2.1);

$$U(M) = \frac{\sqrt[n]{x}}{n-1} + \left(\frac{\sqrt[n]{x}}{\zeta(2)} - 1\right) + \left(1 + \frac{1}{\zeta(2)}\right)\sum_{i=1}^{M} x^{\frac{1}{n2^i}} + 2x^{\frac{1}{n2^{M+1}}}.$$

Now, we have

$$\Delta U(M) := U(M) - U(M+1) = \left(1 - \frac{1}{\zeta(2)}\right) x^{\frac{1}{n2^{M+1}}} - 2x^{\frac{1}{n2^{M+2}}}. \tag{2.2}$$

For $x > \left(\frac{2}{1-\frac{1}{\zeta(2)}}\right)^{n2^{M+1}}$ we easily have $\Delta U(M) > 0$; which means that (2.1) is self sharpening. Also, according to (2.2), we have

$$\Delta U(M) = O\left(x^{\frac{1}{n2^{M+1}}}\right),$$

which is the order of sharpening. □

As you see, the idea of sharpening (1.1) and (2.1) comes from noting the definition of the Mobius function and applying them for $n = 2$.

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References

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