

# INEQUALITIES OF SOME TRIGONOMETRIC FUNCTIONS

CHAO-PING CHEN AND FENG QI

ABSTRACT. By using two identities and two inequalities relating to Bernoulli's and Euler's numbers and power series expansions of cotangent function, secant function, cosecant function and logarithms of functions involving sine function, cosine function and tangent function, six inequalities involving tangent function, cotangent function, sine function, secant function and cosecant function are established.

## 1. INTRODUCTION

The Bernoulli's numbers  $B_n$  and Euler's numbers  $E_n$  for nonnegative integers  $n$  are respectively defined in [1, 6] and [28, p. 1 and p. 6] by

$$\frac{t}{e^t - 1} + \frac{t}{2} = 1 + \sum_{n=0}^{\infty} (-1)^{n-1} B_n \frac{t^{2n}}{(2n)!}, \quad |t| < 2\pi \quad (1)$$

and

$$\frac{2e^{t/2}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n E_n}{(2n)!} \left(\frac{t}{2}\right)^{2n}, \quad |t| < \pi. \quad (2)$$

The following power series expansions are well known and can be found in [1] and [6, pp. 227–229]:

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} B_k}{(2k)!} x^{2k-1}, \quad 0 < |x| < \pi, \quad (3)$$

$$\sec x = \sum_{k=0}^{\infty} \frac{E_k}{(2k)!} x^{2k}, \quad |x| < \frac{\pi}{2}, \quad (4)$$

$$\csc x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)B_k}{(2k)!} x^{2k-1}, \quad 0 < |x| < \pi, \quad (5)$$

$$\ln \frac{\sin x}{x} = - \sum_{k=1}^{\infty} \frac{2^{2k-1} B_k}{k(2k)!} x^{2k}, \quad 0 < |x| < \pi, \quad (6)$$

---

2000 *Mathematics Subject Classification.* 26D05.

*Key words and phrases.* inequality, power series expansion, tangent function, cotangent function, secant function, cosecant function, sine function, cosine function, Bernoulli's number, Euler's number.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), Doctor Fund of Jiaozuo Institute of Technology, CHINA.

This paper was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ .

$$\ln \cos x = - \sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)B_k}{k(2k)!} x^{2k}, \quad |x| < \frac{\pi}{2}, \quad (7)$$

$$\ln \frac{\tan x}{x} = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k-1}-1)B_k}{k(2k)!} x^{2k}, \quad 0 < |x| < \frac{\pi}{2}. \quad (8)$$

The following inequalities relating to Bernoulli's numbers and Euler's numbers are given in [1, p. 805] and [11, p. 421]:

$$\frac{2(2n)!}{(2\pi)^{2n}} < B_n < \frac{2^{2n-1}}{2^{2n-1}-1} \cdot \frac{2(2n)!}{(2\pi)^{2n}}, \quad (9)$$

$$\frac{2^{2(n+1)}(2n)!}{\pi^{2n+1}} > E_n > \frac{3^{2n+1}}{1+3^{2n+1}} \cdot \frac{2^{2(n+1)}(2n)!}{\pi^{2n+1}}. \quad (10)$$

It is also well known [6, p. 231] that

$$\sum_{m=1}^{\infty} \frac{1}{m^{2n}} = \frac{\pi^{2n} 2^{2n-1}}{(2n)!} B_n. \quad (11)$$

The Becker-Stark's inequality ([2], [17, p. 156] and [11, p. 351]) states that for  $0 < x < 1$ ,

$$\frac{4}{\pi} \cdot \frac{x}{1-x^2} < \tan \frac{\pi x}{2} < \frac{\pi}{2} \cdot \frac{x}{1-x^2}. \quad (12)$$

For  $x \in (0, \frac{\pi}{6})$ , Djokvie's inequality states [11, p. 350] that

$$x + \frac{1}{3}x^3 < \tan x < x + \frac{4}{9}x^3. \quad (13)$$

In [3], the following inequalities are proved: For  $x \in (0, \frac{\pi}{2})$  and  $n \in \mathbb{N}$ ,

$$\frac{2^{2(n+1)}(2^{2(n+1)}-1)B_{n+1}}{(2n+2)!} x^{2n} \tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n} \tan x, \quad (14)$$

where

$$S_n(x) = \sum_{i=1}^n \frac{2^{2i}(2^{2i}-1)B_i}{(2i)!} x^{2i-1}. \quad (15)$$

If taking  $n = 1$  in (14), for  $0 < x < \frac{3}{\pi} \sqrt{\frac{5(\pi^2-8)}{38}}$ , the left hand side inequality in (14) is better than the left hand side inequality in (12). If taking  $n = 2$  in (14), we obtain

$$x + \frac{1}{3}x^3 + \frac{2}{15}x^4 \tan x < \tan x < x + \frac{1}{3}x^3 + \left(\frac{2}{\pi}\right)^4 x^4 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right). \quad (16)$$

The constants  $\frac{2}{15}$  and  $\left(\frac{2}{\pi}\right)^4$  in (16) are the best possible. Since

$$\frac{1}{3} + \left(\frac{2}{\pi}\right)^4 x \tan x < \frac{1}{3} + \left(\frac{2}{\pi}\right)^4 \cdot \frac{\pi}{6} \cdot \frac{1}{\sqrt{3}} < \frac{4}{9},$$

the inequalities in (16) are better than those in (13).

In recent years, there is a amounts of literature on inequalities involving trigonometric functions [4, 5, 7, 8, 10, 19, 20, 23], estimates of remainders of elementary functions [16, 18] and related questions [21, 24].

The purpose of this paper is to prove the following six inequalities of some trigonometric functions.

**Theorem 1.** For  $0 < x < 1$ ,

$$\frac{2}{\pi} \cdot \frac{x}{1-x^2} < \frac{1}{\pi x} - \cot(\pi x) < \frac{\pi}{3} \cdot \frac{x}{1-x^2}, \quad (17)$$

$$\frac{\pi^2}{8} \cdot \frac{x}{1-x^2} < \sec \frac{\pi x}{2} - 1 < \frac{4}{\pi} \cdot \frac{x}{1-x^2}, \quad (18)$$

$$\frac{\pi}{6} \cdot \frac{x}{1-x^2} < \csc(\pi x) - \frac{1}{\pi x} < \frac{2}{\pi} \cdot \frac{x}{1-x^2}. \quad (19)$$

The constants  $\frac{2}{\pi}$  and  $\frac{\pi}{3}$  in (17),  $\frac{\pi^2}{8}$  and  $\frac{4}{\pi}$  in (18),  $\frac{\pi}{6}$  and  $\frac{2}{\pi}$  in (19) are the best possible.

For  $0 < |x| < 1$ , we have

$$\ln \left( \frac{\pi x}{\sin(\pi x)} \right) < \frac{\pi^2}{6} \cdot \frac{x^2}{1-x^2}, \quad (20)$$

$$\ln \left( \sec \frac{\pi x}{2} \right) < \frac{\pi^2}{8} \cdot \frac{x^2}{1-x^2}, \quad (21)$$

$$\ln \left( \frac{\tan \frac{\pi x}{2}}{\frac{\pi x}{2}} \right) < \frac{\pi^2}{12} \cdot \frac{x^2}{1-x^2}. \quad (22)$$

The constants  $\frac{\pi^2}{6}$ ,  $\frac{\pi^2}{8}$  and  $\frac{\pi^2}{12}$  are the best possible.

*Remark 1.* Notice that there are a large number of particular inequalities relating to trigonometric functions in [11, 17].

## 2. PROOF OF THEOREM 1

The first roof of inequality (17). Define for  $0 < x < 1$

$$f(x) = \frac{1-x^2}{x} \left( \frac{1}{\pi x} - \cot(\pi x) \right). \quad (23)$$

Replacing  $x$  by  $\pi x$  in (3) yields

$$\cot(\pi x) = \frac{1}{\pi x} - \sum_{k=1}^{\infty} \frac{2^{2k} \pi^{2k-1} B_k}{(2k)!} x^{2k-1}, \quad 0 < |x| < 1. \quad (24)$$

Substituting (24) into (23) produces

$$f(x) = \frac{\pi}{3} + \sum_{k=1}^{\infty} \left( \frac{2^{2k+2} \pi^{2k+1} B_{k+1}}{(2k+2)!} - \frac{2^{2k} \pi^{2k-1} B_k}{(2k)!} \right) x^{2k}. \quad (25)$$

Using (11), (25) can be rewritten as

$$f(x) = \frac{\pi}{3} - \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{n^{2k}} - \frac{1}{n^{2k+2}} \right) x^{2k}.$$

It is easy to see that  $f(x)$  is strictly decreasing, then

$$\frac{2}{\pi} = \lim_{x \rightarrow 1} f(x) < f(x) < \lim_{x \rightarrow 0} f(x) = \frac{\pi}{3}.$$

Inequality (17) follows.  $\square$

The second proof of inequality (17). The following inequalities are deduced from (9):

$$\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} > \frac{2}{\pi}, \quad k \geq 1, \quad (26)$$

$$\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} < \frac{2}{\pi} \cdot \frac{2^{2k-1}}{2^{2k-1}-1} < \frac{\pi}{3}, \quad k \geq 2. \quad (27)$$

Replacing  $x$  by  $\pi x$  in (3) and using (26), we see that for  $0 < x < 1$ ,

$$\frac{1}{\pi x} - \cot(\pi x) = \sum_{k=1}^{\infty} \frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} x^{2k-1} > \frac{2}{\pi} \sum_{k=1}^{\infty} x^{2k-1} = \frac{2}{\pi} \cdot \frac{x}{1-x^2}. \quad (28)$$

Similarly, by using (27), we have for  $0 < x < 1$ ,

$$\begin{aligned} \frac{1}{\pi x} - \cot(\pi x) &= \sum_{k=1}^{\infty} \frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} x^{2k-1} = \frac{\pi}{3}x + \sum_{k=2}^{\infty} \frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} x^{2k-1} \\ &< \frac{\pi}{3}x + \frac{\pi}{3} \sum_{k=2}^{\infty} x^{2k-1} = \frac{\pi}{3} \sum_{k=1}^{\infty} x^{2k-1} = \frac{\pi}{3} \cdot \frac{x}{1-x^2}. \end{aligned} \quad (29)$$

From L'Hospital rule, it follows that

$$\lim_{x \rightarrow 0^+} \frac{1-x^2}{x} \left( \frac{1}{\pi x} - \cot(\pi x) \right) = \frac{\pi}{3}, \quad (30)$$

$$\lim_{x \rightarrow 1^-} \frac{1-x^2}{x} \left( \frac{1}{\pi x} - \cot(\pi x) \right) = \frac{2}{\pi}. \quad (31)$$

Thus, the constants  $\frac{2}{\pi}$  and  $\frac{\pi}{3}$  in (17) are the best possible.  $\square$

*Proof of inequality (18).* The following inequalities follow from (10):

$$\frac{E_n}{(2n)!} \left( \frac{\pi}{2} \right)^{2n} < \frac{4}{\pi}, \quad n \geq 1, \quad (32)$$

$$\frac{E_n}{(2n)!} \left( \frac{\pi}{2} \right)^{2n} > \frac{4}{\pi} \cdot \frac{3^{2n+1}}{3^{2n+1}+1} > \frac{\pi^2}{8}, \quad n \geq 2. \quad (33)$$

Replacing  $x$  by  $\frac{\pi x}{2}$  in (4) and using (32), we obtain that for  $0 < |x| < 1$ ,

$$\sec \frac{\pi x}{2} - 1 = \sum_{n=1}^{\infty} \frac{E_n}{(2n)!} \left( \frac{\pi}{2} \right)^{2n} x^{2n} < \frac{4}{\pi} \sum_{n=1}^{\infty} x^{2n} = \frac{4}{\pi} \cdot \frac{x^2}{1-x^2}, \quad (34)$$

and, by using (33), we have for  $0 < |x| < 1$ ,

$$\begin{aligned} \sec \frac{\pi x}{2} - 1 &= \sum_{n=1}^{\infty} \frac{E_n}{(2n)!} \left( \frac{\pi}{2} \right)^{2n} x^{2n} = \frac{\pi^2}{8}x^2 + \sum_{n=2}^{\infty} \frac{E_n}{(2n)!} \left( \frac{\pi}{2} \right)^{2n} x^{2n} \\ &> \frac{\pi^2}{8}x^2 + \frac{\pi^2}{8} \sum_{n=2}^{\infty} x^{2n} = \frac{\pi^2}{8} \sum_{n=1}^{\infty} x^{2n} = \frac{\pi^2}{8} \cdot \frac{x^2}{1-x^2}. \end{aligned} \quad (35)$$

Further, since

$$\lim_{x \rightarrow 0^+} \frac{1-x^2}{x^2} \left[ \sec \frac{\pi x}{2} - 1 \right] = \frac{\pi^2}{8}, \quad (36)$$

$$\lim_{x \rightarrow 1^-} \frac{1-x^2}{x^2} \left[ \sec \frac{\pi x}{2} - 1 \right] = \frac{4}{\pi}, \quad (37)$$

the constants  $\frac{\pi^2}{8}$  and  $\frac{4}{\pi}$  in (18) are the best possible.  $\square$

*Proof of inequality (19).* The following inequalities can be deduced from (9):

$$\frac{2(2^{2n-1} - 1)\pi^{2n-1}B_n}{(2n)!} < \frac{2}{\pi}, \quad n \geq 1, \quad (38)$$

$$\frac{2(2^{2n-1} - 1)\pi^{2n-1}B_n}{(2n)!} > \frac{2}{\pi} \cdot \frac{2^{2n-1} - 1}{2^{2n-1}} > \frac{\pi}{6}, \quad n \geq 2. \quad (39)$$

Replacing  $x$  by  $\pi x$  in (5) and using (38) gives that for  $0 < x < 1$ ,

$$\csc(\pi x) - \frac{1}{\pi x} = \sum_{n=1}^{\infty} \frac{2(2^{2n-1} - 1)\pi^{2n-1}B_n}{(2n)!} x^{2n-1} < \frac{2}{\pi} \sum_{n=1}^{\infty} x^{2n-1} = \frac{2}{\pi} \cdot \frac{x}{1-x^2}, \quad (40)$$

and, employing (39) yields that for  $0 < x < 1$ ,

$$\begin{aligned} \csc(\pi x) - \frac{1}{\pi x} &= \sum_{n=1}^{\infty} \frac{2(2^{2n-1} - 1)\pi^{2n-1}B_n}{(2n)!} x^{2n-1} \\ &= \frac{\pi}{6}x + \sum_{n=2}^{\infty} \frac{2(2^{2n-1} - 1)\pi^{2n-1}B_n}{(2n)!} x^{2n-1} \\ &> \frac{\pi}{6}x + \frac{\pi}{6} \sum_{n=2}^{\infty} x^{2n-1} = \frac{\pi}{6} \sum_{n=1}^{\infty} x^{2n-1} = \frac{\pi}{6} \cdot \frac{x}{1-x^2}. \end{aligned} \quad (41)$$

It is easy to see that

$$\lim_{x \rightarrow 0^+} \frac{1-x^2}{x} \left( \csc(\pi x) - \frac{1}{\pi x} \right) = \frac{\pi}{6}, \quad (42)$$

$$\lim_{x \rightarrow 1^-} \frac{1-x^2}{x} \left( \csc(\pi x) - \frac{1}{\pi x} \right) = \frac{2}{\pi}. \quad (43)$$

Therefore, the constants  $\frac{\pi}{6}$  and  $\frac{2}{\pi}$  in (19) are the best possible.  $\square$

*The first proof of inequality (20).* Define for  $0 < x < 1$

$$g(x) = \frac{1-x^2}{x^2} \ln \frac{\pi x}{\sin(\pi x)}. \quad (44)$$

Replacing  $x$  by  $\pi x$  in (6) yields

$$\ln \frac{\pi x}{\sin(\pi x)} = \sum_{k=1}^{\infty} \frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} x^{2k}, \quad 0 < |x| < 1. \quad (45)$$

Substituting (45) into (44) leads to

$$g(x) = \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \left( \frac{2^{2k+1}\pi^{2k+2}B_{k+1}}{(k+1)(2k+2)!} - \frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} \right) x^{2k}. \quad (46)$$

Using (11), (46) can be rearranged to

$$g(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \right) x^{2k}.$$

It is easy to see that  $g(x)$  is strictly decreasing, thus

$$g(x) < \lim_{x \rightarrow 0} g(x) = \frac{\pi^2}{6},$$

which is equivalent to (20).  $\square$

*The second proof of inequality (20).* It follows from (9) that

$$\frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} < \frac{2^{2k-1}}{k(2^{2k-1}-1)} < \frac{\pi^2}{6}, \quad k \geq 2. \quad (47)$$

Replacing  $x$  by  $\pi x$  in (6) and using (47), we obtain for  $0 < |x| < 1$ ,

$$\begin{aligned} \ln \frac{\pi x}{\sin(\pi x)} &= \sum_{k=1}^{\infty} \frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} x^{2k} = \frac{\pi^2}{6} x^2 + \sum_{k=2}^{\infty} \frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} x^{2k} \\ &< \frac{\pi^2}{6} x^2 + \frac{\pi^2}{6} \sum_{k=2}^{\infty} x^{2k} = \frac{\pi^2}{6} \sum_{k=1}^{\infty} x^{2k} = \frac{\pi^2}{6} \cdot \frac{x^2}{1-x^2}. \end{aligned} \quad (48)$$

Since

$$\lim_{x \rightarrow 0^+} \frac{1-x^2}{x^2} \ln \frac{\pi x}{\sin(\pi x)} = \frac{\pi^2}{6}, \quad (49)$$

the constant  $\frac{\pi^2}{6}$  in (20) is the best possible.  $\square$

*The first proof of inequality (21).* Define for  $0 < x < 1$

$$h(x) = \frac{1-x^2}{x^2} \ln \left( \sec \frac{\pi x}{2} \right). \quad (50)$$

Replacing  $x$  by  $\frac{\pi x}{2}$  in (7) yields

$$\ln \left( \sec \frac{\pi x}{2} \right) = \sum_{k=1}^{\infty} \frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!} x^{2k}, \quad 0 < |x| < 1. \quad (51)$$

Substituting (51) into (50) leads to

$$h(x) = \frac{\pi^2}{8} - \sum_{k=1}^{\infty} \left( \frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!} - \frac{(2^{2k+2}-1)\pi^{2k+2}B_{k+1}}{(2k+2)(2k+2)!} \right) x^{2k}. \quad (52)$$

Using (11), (52) can be rewritten as

$$h(x) = \frac{\pi^2}{8} - \sum_{k=1}^{\infty} \left( \frac{2^{2k}-1}{k2^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \frac{2^{2k+2}-1}{(k+1)2^{2k+2}} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \right) x^{2k}. \quad (53)$$

It is clear that for  $k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} > \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \quad (54)$$

and

$$\frac{2^{2k}-1}{k2^{2k}} > \frac{2^{2k+2}-1}{(k+1)2^{2k+2}}. \quad (55)$$

From (53), (54) and (55), we readily obtain that  $h(x)$  is strictly decreasing. Thus

$$g(x) < \lim_{x \rightarrow 0} g(x) = \frac{\pi^2}{8}, \quad (56)$$

which is equivalent to (21).  $\square$

The second proof of inequality (21). It follows from (9) that

$$\frac{(2^{2k} - 1)\pi^{2k} B_k}{2k(2k)!} < \frac{2^{2k} - 1}{k(2^{2k} - 2)} < \frac{\pi^2}{8}, \quad k \geq 2. \quad (57)$$

Replacing  $x$  by  $\frac{\pi x}{2}$  in (7) and using (57), we have for  $0 < |x| < 1$ ,

$$\begin{aligned} \ln \left( \sec \frac{\pi x}{2} \right) &= \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)\pi^{2k} B_k}{2k(2k)!} x^{2k} = \frac{\pi^2}{8} x^2 + \sum_{k=2}^{\infty} \frac{(2^{2k} - 1)\pi^{2k} B_k}{2k(2k)!} x^{2k} \\ &< \frac{\pi^2}{8} x^2 + \frac{\pi^2}{8} \sum_{k=2}^{\infty} x^{2k} = \frac{\pi^2}{8} \sum_{k=1}^{\infty} x^{2k} = \frac{\pi^2}{8} \cdot \frac{x^2}{1 - x^2}. \end{aligned} \quad (58)$$

It is clear that

$$\lim_{x \rightarrow 0^+} \frac{1 - x^2}{x^2} \ln \left( \sec \frac{\pi x}{2} \right) = \frac{\pi^2}{8}. \quad (59)$$

Thus, the constant  $\frac{\pi^2}{8}$  in (21) is the best possible.  $\square$

The first proof of inequality (22). Define for  $0 < x < 1$

$$\varphi(x) = \frac{1 - x^2}{x^2} \ln \left( \frac{\tan \frac{\pi x}{2}}{\frac{\pi x}{2}} \right). \quad (60)$$

Replacing  $x$  by  $\frac{\pi x}{2}$  in (8) yields

$$\ln \left( \frac{\tan \frac{\pi x}{2}}{\frac{\pi x}{2}} \right) = \sum_{k=1}^{\infty} \frac{(2^{2k-1} - 1)\pi^{2k} B_k}{k(2k)!} x^{2k}, \quad 0 < |x| < 1. \quad (61)$$

Substituting (61) into (60) gives

$$\varphi(x) = \frac{\pi^2}{12} - \sum_{k=1}^{\infty} \left( \frac{(2^{2k-1} - 1)\pi^{2k} B_k}{k(2k)!} - \frac{(2^{2k+1} - 1)\pi^{2k+2} B_{k+1}}{(k+1)(2k+2)!} \right) x^{2k}. \quad (62)$$

Using (11), (62) can be rewritten as

$$\varphi(x) = \frac{\pi^2}{12} - \sum_{k=1}^{\infty} \left( \frac{2^{2k-1} - 1}{k 2^{2k-1}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \frac{2^{2k+1} - 1}{(k+1) 2^{2k+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \right) x^{2k}. \quad (63)$$

Combining (54) and (55) with (63), we see that  $\varphi(x)$  is strictly decreasing. Hence

$$\varphi(x) < \lim_{x \rightarrow 0} \varphi(x) = \frac{\pi^2}{12}, \quad (64)$$

which is equivalent to (22).  $\square$

The second proof of inequality (22). The following inequality is deduced from (9):

$$\frac{(2^{2k-1} - 1)\pi^{2k} B_k}{k(2k)!} < \frac{1}{k} < \frac{\pi^2}{12}, \quad k \geq 2. \quad (65)$$

Replacing  $x$  by  $\frac{\pi x}{2}$  in (8) and using (65), we have for  $0 < |x| < 1$ ,

$$\begin{aligned} \ln \left( \tan \frac{\pi x}{2} / \frac{\pi x}{2} \right) &= \sum_{k=1}^{\infty} \frac{(2^{2k-1} - 1)\pi^{2k} B_k}{k(2k)!} x^{2k} \\ &= \frac{\pi^2}{12} x^2 + \sum_{k=2}^{\infty} \frac{(2^{2k-1} - 1)\pi^{2k} B_k}{k(2k)!} x^{2k} \end{aligned}$$

$$< \frac{\pi^2}{12}x^2 + \frac{\pi^2}{12} \sum_{k=2}^{\infty} x^{2k} = \frac{\pi^2}{12} \sum_{k=1}^{\infty} x^{2k} = \frac{\pi^2}{12} \frac{x^2}{1-x^2}. \quad (66)$$

Direct computing yields

$$\lim_{x \rightarrow 0^+} \frac{1-x^2}{x^2} \ln \left( \frac{\tan \frac{\pi x}{2}}{\frac{\pi x}{2}} \right) = \frac{\pi^2}{12}. \quad (67)$$

Thus, the constant  $\frac{\pi^2}{12}$  in (22) is the best possible.  $\square$

*Remark 2.* Motivated by ideas in [27], Bernoulli's numbers and polynomials and Euler's numbers and polynomials are generalized or extended and basic properties and recurrence formulas of them are established in [9, 12, 13, 14, 15, 22, 25, 26] step by step.

#### REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, Washington, 1965, 1972.
- [2] M. Becker and E. L. Stark, *On a hierarchy of quolynomial inequalities for  $\tan x$* , Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **602-633** (1978), 133–138.
- [3] Ch.-P. Chen and F. Qi, *A double inequality for remainder of power series of tangent function*, Tamkang J. Math. **34** (2003), no. 3, accepted. RGMIA Res. Rep. Coll. **5** (2002), suppl., Art. 2. Available online at [http://rgmia.vu.edu.au/v5\(E\).html](http://rgmia.vu.edu.au/v5(E).html).
- [4] Ch.-P. Chen and F. Qi, *On two new proofs of Wilker's inequality*, Studies in College Mathematics (Gāoděng Shùxué Yānjiū) **5** (2002), no. 4, 38–39. (Chinese)
- [5] Ch.-P. Chen and F. Qi, *Three inequalities involving hyperbolically trigonometric functions*, RGMIA Res. Rep. Coll. **6** (2003). Available online at <http://rgmia.vu.edu.au>.
- [6] Group of compilation, *Handbook of Mathematics*, Peoples' Education Press, Beijing, China, 1979. (Chinese)
- [7] B.-N. Guo, W. Li and F. Qi, *Proofs of Wilker's inequalities involving trigonometric functions*, Inequality Theory and Applications **2** (2003), in press. Y. J. Cho, J. K. Kim and S. S. Dragomir (Ed.), Nova Science Publishers.
- [8] B.-N. Guo, W. Li and F. Qi, *On new proofs of inequalities involving trigonometric functions*, RGMIA Res. Rep. Coll. **3** (2000), no. 1, Art. 15, 167–170. Available online at <http://rgmia.vu.edu.au/v3n1.html>.
- [9] B.-N. Guo and F. Qi, *Generalisation of Bernoulli polynomials*, Internat. J. Math. Ed. Sci. Tech. **33** (2002), no. 3, 428–431.
- [10] B.-N. Guo, B.-M. Qiao, F. Qi and W. Li, *On new proofs of Wilker's inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), no. 1, 19–22.
- [11] J.-Ch. Kuang, *Chángyòng Bùděngshì (Applied Inequalities)*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)
- [12] Q.-M. Luo, B.-N. Guo, and F. Qi, *Generalizations of Bernoulli numbers and polynomials*, Internat. J. Math. Math. Sci. (2003), in press. RGMIA Res. Rep. Coll. **5** (2002), no. 2, Art. 12, 353–359. Available online at <http://rgmia.vu.edu.au/v5n2.html>.
- [13] Q.-M. Luo and F. Qi, *Generalizations of Euler numbers and polynomials*, Internat. J. Math. Math. Sci. (2003), accepted. RGMIA Res. Rep. Coll. **5** (2002), suppl., Art. 4. Available online at [http://rgmia.vu.edu.au/v5\(E\).html](http://rgmia.vu.edu.au/v5(E).html).
- [14] Q.-M. Luo and F. Qi, *Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **6** (2003), no. 2, in press. RGMIA Res. Rep. Coll. **5** (2002), no. 3, Art. 1, 405–412. Available online at <http://rgmia.vu.edu.au/v5n3.html>.
- [15] Q.-M. Luo, F. Qi, and B.-N. Guo, *Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials of higher orders*, submitted.
- [16] M. Merkle, *Inequalities for residuals of power series: a review*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **6** (1995), 79–85.
- [17] J. Pečarić, *Nejednakosti*, Element, Zagreb, 1996. (Croatia)



- [18] F. Qi, *A method of constructing inequalities about  $e^x$* , Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 16–23.
- [19] F. Qi, *Refinements and extensions of Jordan's and Kober's inequalities*, Gōngkē Shùxué (Mathematics for Technology) **12** (1996), no. 4, 98–102. (Chinese)
- [20] F. Qi, L.-H. Cui and S.-L. Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (1999), no. 4, 517–528.
- [21] F. Qi and B.-N. Guo, *Estimate for upper bound of an elliptic integral*, Math. Practice Theory **26** (1996), no. 3, 285–288. (Chinese)
- [22] F. Qi and B.-N. Guo, *Generalized Bernoulli polynomials*, RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 10, 691–695. Available online at <http://rgmia.vu.edu.au/v4n4.html>.
- [23] F. Qi and Q.-D. Hao, *Refinements and sharpenings of Jordan's and Kober's inequality*, Math. Inform. Quart. **8** (1998), no. 3, 116–120.
- [24] F. Qi and Zh. Huang, *Inequalities of the complete elliptic integrals*, Tamkang J. Math. **29** (1998), no. 3, 165–169.
- [25] F. Qi and Q.-M. Luo, *Generalized Euler's numbers and polynomials of higher order*, submitted.
- [26] F. Qi, Q.-M. Luo, and B.-N. Guo, *Generalized Bernoulli's numbers and polynomials of higher order*, submitted.
- [27] F. Qi and S.-L. Xu, *The function  $(b^x - a^x)/x$ : Inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359. Available online at <http://www.ams.org/journal-getitem?pii=S0002-9939-98-04442-6>.
- [28] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn (Introduction to Special Function)*, The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

(Ch.-P. Chen) DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, JIAOZUO CITY, HENAN 454000, CHINA

(F. Qi) DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS, JIAOZUO INSTITUTE OF TECHNOLOGY, JIAOZUO CITY, HENAN 454000, CHINA

*E-mail address:* qifeng@jzhit.edu.cn, fengqi618@member.ams.org

*URL:* <http://rgmia.vu.edu.au/qi.html>