Variational inequalities and monotone trajectories of differential inclusions

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Abstract

The main task of this paper is to study the links between solutions of Variational Inequalities and monotonicity of the trajectories of a special kind of Differential Inclusions (namely "projected differential inclusions"). The case in which the involved operators are single-valued has been considered in[18], where the connections between solutions of variational inequalities and stability of the solutions of a "projected differential equation" have been investigated.

Key words: Differential inclusions, variational inequalities, monotone trajectories.

1 Introduction

The relations of Minty and Stampacchia Variational Inequalities with differentiable optimization problems have been widely studied. Basically it has been proved that Stampacchia Variational Inequality (for short, SVI) is a necessary condition for optimality (see e.g. [13]), while Minty Variational Inequality (for short, MVI) is a sufficient one (see e.g. [7],[14]). Generalizations of SVI and MVI to point to set maps have been introduced (see e.g. [4], [9]) and the previous results have been proved also for non differentiable optimization problems (see e.g. [5]).

On the other hand Dynamical Systems (for short, DS) are a classical tool for dealing with a wide range both of real and mathematical problems. Recently the existence and stability of equilibria of a (projected) DS have been characterized by means of variational inequalities. In this context it has been proved that existence of a solution of SVI is equivalent to existence of an equilibrium, while MVI ensures the stability of equilibria (see for instance [18]).
Therefore, we prove that variational inequalities for point to set maps can be related to differential inclusions. The main task of this paper is to study the links between solutions of MVI and monotonicity of the trajectories of a special kind of Differential Inclusions (namely "projected differential inclusions"). The case in which the involved operators are single-valued has been considered in [18], where the connections between solutions of variational inequalities and stability of the solutions of a "projected differential equation" have been investigated.
The paper is organized as follows. In Section 2 we recall the main concepts and results about Differential Inclusions and Variational Inequalities. Section 3 states the links between existence of solutions of a Variational Inequality of Minty type and monotonicity of trajectories of a Differential Inclusion. Section 4 presents some necessary and sufficient conditions for the existence

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of monotone trajectories of a classical differential equation, when the involved operator is locally Lipschitz. These results generalize analogous results given in [18], in the case in which the operator is of class C^1. Finally, Section 5 presents an application of the previous results to generalized gradient inclusions.

2 Preliminaries

2.1 Differential Inclusions

Let K be a closed convex subset of $\mathbb{R}^n$.

Theorem 1. *[1] We can associate to every $x \in \mathbb{R}^n$ a unique element $\pi_K(x) \in K$, satisfying:*

$$\|x - \pi_K(x)\| = \min_{y \in K} \|x - y\|.$$

It is characterized by the following inequality:

$$\langle \pi_K(x) - x, \pi_K(x) - y \rangle \leq 0, \quad \forall y \in K,$$

and it is non esxpansive, i.e.:

$$\|\pi_K(x) - \pi_K(y)\| \leq \|x - y\|.$$

The map π_K is said the projector (of best approximation) onto K. When K is a linear subspace, then π_K is linear (see [1]). We set $\pi_K(0) = m(K)$ (i.e. $m(K)$ denotes the element of K with minimal norm).
We denote by:

$$C^- = \{v \in \mathbb{R}^n : \langle v, a \rangle \leq 0, \forall a \in C\}$$

the (negative) polar cone of the set $C \subseteq \mathbb{R}^n$, while:

$$T(C, x) = \{v \in \mathbb{R}^n : \exists v_n \to v,\ \alpha_n > 0,\ \alpha_n \to 0,\ x + \alpha_n v_n \in C\}$$

is Bouligand tangent cone to the set C at $x \in \mathrm{cl}C$ and $N(C, x) = [T(C, x)]^-$ stands for the normal cone to C at $x \in \mathrm{cl}C$.
It is known that $T(C, x)$ and $N(C, x)$ are closed sets. Furthermore, when we consider a closed convex set $K \subseteq \mathbb{R}^n$, then $T(K, x) = \mathrm{cl}\,\mathrm{cone}\,(K - x)$ ($\mathrm{cone}\,A$ denotes the cone generated by the set A), so that both the tangent cone and the normal cone are also convex.

Proposition 1. *Let A be a compact convex subset of $\mathbb{R}^n$, T be a closed convex cone and $N = T^-$ be its polar cone. Then:*

$$\pi_T(A) \subseteq A - N. \tag{1}$$

The elements of minimal norm are equal:

$$m(\pi_T(A)) = m(A - N)$$

and satisfy:

$$\sup_{z \in -A} \langle z, m(\pi_T(A)) \rangle + \|m(\pi_T(A)\|^2 \leq 0.$$

We recall that, given a map $G : K \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$, a differential inclusion is the problem of finding an absolutely continuous function $x(\cdot)$, defined on an interval $[0, T]$, such that:

$$\begin{cases} \forall t \in [0,T], \quad x(t) \in K, \\ \text{for a.a. } t \in [0,T], \quad x'(t) \in G(x(t)). \end{cases}$$

The solutions of the previous problem are called also "trajectories" of the differential inclusion. If $x(\cdot)$ is such that:

$$\begin{cases} \forall t \in [0,T], \quad x(t) \in K, \\ \text{for a.a. } t \in [0,T], \quad x'(t) = m(G(x(t))) \end{cases}$$

then it is called a slow solution of the differential inclusion.

Definition 1. *A map F from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$ is said to be upper semicontinuous (u.s.c.) at $x_0 \in \mathbb{R}^n$, when for every open set N containing x_0, there exists a neighborhood M of x_0 such that $F(M) \subseteq N$.*
F is said to be u.s.c. when it is so at every $x_0 \in \mathbb{R}^n$.

From now on, if not othewise specified we will do the following:

Standing Assumptions

i) K will denote a convex and closed subset of $\mathbb{R}^n$;

ii) F will denote an u.s.c. map from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$, with nonempty convex and compact values.

We are concerned with the following problem, which is a special case of differential inclusion.

Problem 1. *Find an absolutely continuous function $x(\cdot)$ from $[0, T]$ into $\mathbb{R}^n$, satisfying:*

$$DVI(F,K) \quad \begin{cases} \forall t \in [0,T], \quad x(t) \in K, \\ \text{for a.a. } t \in [0,T], \quad x'(t) \in -F(x(t)) - N(K, x(t)) \end{cases}$$

The previous problem is called a "differential variational inequality" (for short, DVI) [1]. The following result states the equivalence of $DVI(F, K)$ and a "projected differential inclusion" (for short, PDI) [1].

Theorem 2. *The solutions of Problem 1 are the solutions of:*

$$PDI(F,K) \quad \begin{cases} \forall t \in [0,T], \quad x(t) \in K, \\ \text{for a.a. } t \in [0,T], \quad x'(t) \in \pi_{T(K,x(t))}\left(-F(x(t)\right), \end{cases}$$

and conversely.

Remark 1. We recall that when F is a single-valued operator, then the corresponding "projected differential equation" and its applications have been studied for instance in [8], [17], [18].

Theorem 3. *[1] The slow solutions of $DVI(F, K)$ and $PDI(F, K)$ coincide.*

Definition 2. *A point $x^* \in K$ is an equilibrium point for $DVI(F, K)$, when:*

$$0 \in -F(x^*) - N(K, x^*).$$

We recall the following existence result.

Theorem 4. *a) If K is compact, then there exists an equlibrium point for $DVI(F,K)$.*

b) If $m(F(\cdot))$ is bounded, then, for any $x_0 \in K$ there exists an absolutely continuous function $x(t)$ defined on an interval $[0,T]$, such that:

$$\begin{cases} x(0) = x_0, \quad x'(t) \in -F(x(t)) - N_K(x(t)) & \text{for a.a. } t \in [0,T] \\ \forall t \in [0,T], \quad x(t) \in K \end{cases}$$

We close this section, recalling the notion of monotonicity of a trajectory of $DVI(F,K)$ [1].

Definition 3. *Let V be a function from K to $\mathbb{R}^+$. A trajectory $x(t)$ of $DVI(F,K)$ is monotone (with respect to V) when:*

$$\forall t \geq s, \quad V(x(t)) - V(x(s)) \leq 0.$$

If the previous inequality holds strictly $\forall t > s$, then we say that $x(t)$ is strictly monotone w.r.t. V.

In the following we will be interested in the trajectories of $DVI(F,K)$ which are monotone w.r.t. the function:

$$\tilde{V}_{x^*}(x) = \frac{\|x - x^*\|^2}{2},$$

where x^* is an equlibrium point of $DVI(F,K)$.

2.2 Variational Inequalities

We consider the following formulations of a variational inequality (see for instance [4], [9], [11]):

Definition 4. *A point $x^* \in K$ is a solution of a Stampacchia Variational Inequality (for short, SVI) when $\exists \xi^* \in F(x^*)$ such that:*

$$SVI(F,K) \quad \langle \xi^*, y - x^* \rangle \geq 0 \ , \ \forall y \in K.$$

Definition 5. *A point $x^* \in K$ is a solution of a Strong Minty Variational Inequality (for short, $SMVI$), when:*

$$SMVI(F,K) \quad \langle \xi, y - x^* \rangle \geq 0 \ , \forall y \in K, \ \forall \xi \in F(y).$$

Definition 6. *A point $x^* \in K$ is a solution of a Weak Minty Variational Inequality (for short, $WMVI$), when $\forall y \in K$, $\exists \xi \in F(y)$ such that:*

$$\langle \xi, y - x^* \rangle \geq 0.$$

Definition 7. *If in Definition 5 (resp. 6), strict inequality holds $\forall y \in K$, $y \neq x^*$, then we say that x^* is a "strict " solution of SMVI(F, K) (resp. of WMVI(F, K)).*

Remark 2. When F is single valued, Definitions 5 and 6 reduce to the classical notion of MVI.

The classical Minty Lemma (see for instance [16]) relates Minty Variational Inequalities and Stampacchia Variational Inequalities, when F is a single valued operator. The following result gives an extension to the case in which F is a point-to-set map. We recall first the following definition (see e.g. [11]).

Definition 8. *F is said:*

i) monotone, if for all $x, y \in K$, we have:

$$\forall u \in F(x), \forall v \in F(y) : \langle v - u, y - x\rangle \geq 0;$$

ii) pseudo-monotone (resp. strictly pseudomonotone), if for all $x, y \in K$ (resp. for all $x, y \in K$ with $y \neq x$) the following implication holds:

$$\exists u \in F(x) : \langle u, y - x\rangle \geq 0 \Rightarrow \forall v \in F(y) : \langle v, y - x\rangle \geq 0;$$

$$(\exists u \in F(x) : \langle u, y - x\rangle \geq 0 \Rightarrow \forall v \in F(y) : \langle v, y - x\rangle > 0)$$

ii) quasi-monotone, if for all $x, y \in K$, we have:

$$\exists u \in F(x) : \langle u, y - x\rangle > 0 \Rightarrow \forall v \in F(y) : \langle v, y - x\rangle \geq 0.$$

Remark 3. The following implications are classical:

$$\begin{array}{c} \text{monotone} \Rightarrow \text{pseudomonotone} \Rightarrow \text{quasimonotone} \\ \Uparrow \\ \text{strictly pseudomonotone.} \end{array}$$

Lemma 1. *i) Let F be u.s.c. and K be nonempty closed and convex. Any $x^* \in K$, which solves $WMVI(F, K)$, it is a solution of $SVI(F, K)$ as well.*

ii) If F is a pseudo-monotone map, any solution of $SVI(F, K)$ also solves $SMVI(F, K)$.

iii) If F is a strictly pseudo-monotone map, any solution of $SVI(F, K)$ is a strict solution of $SMVI(F, K)$.

Proof:

i) Let z be an arbitrary point in K and consider $y = x^* + t(z - x^*) \in K$, where $t \in (0, 1)$. Since x^* solves $WMVI(F, K)$, we have that $\forall t \in (0, 1)$, $\exists \xi = \xi(t) \in F(x^* + t(z - x^*))$, such that:

$$\langle \xi(t), t(z - x^*)\rangle \geq 0,$$

that is:

$$\langle \xi(t), z - x^*\rangle \geq 0.$$

Since F is u.s.c., we get that for any integer $n > 0$, there exists a number $\delta_n > 0$ such that, for $t \in (0, \delta_n]$ it holds:

$$F\big(x^* + t(z - x^*)\big) \subseteq F(x^*) + \frac{1}{n}B.$$

Hence, for $t \in (0, \delta_n]$, $\xi(t) = f(t) + \gamma(t)$, where $f(t) \in F(x^*)$ and $\gamma(t) \in \frac{1}{n}B$. Without loss of generality we can assume $\delta_n < 1$ $\forall n$ and we have:

$$0 \leq \langle \xi(t), z - x^*\rangle = \langle f(t), z - x^*\rangle + \langle \gamma(t), z - x^*\rangle.$$

Furthermore, by Cauchy-Schwarz inequality, we get:

$$|\langle \gamma(t), z - x^*\rangle| \leq \|\gamma(t)\| \, \|z - x^*\| \leq \frac{1}{n}\|z - x^*\|,$$

so that, choosing in particular, $t = \delta_n$, we obtain:

$$\langle f(\delta_n), z - x^* \rangle \geq -\frac{1}{n} \|z - x^*\|.$$

Recalling that $F(x^*)$ is a compact set, when $n \to +\infty$ we can assume that $f(\delta_n) \to \bar{f} \in F(x^*)$ and we get:

$$\langle \bar{f}, z - x^* \rangle \geq 0. \tag{2}$$

By the former construction, we have that $\forall z \in K$, there exists $\bar{f} = \bar{f}(z) \in F(x^*)$ such that (2) holds.
Since F is convex and compact-valued, then, from Lemma 1 in Blum and Oettli [3], we get the thesis.

The proof of ii) and iii) is trivial. □

Remark 4. Since every solution of $SMVI(F,K)$ is also a solution of $WMVI(F,K)$, then, from the previous theorem we obtain that, if F is pseudo-monotone, the solution sets of $WMVI(F,K)$, $SMVI(F,K)$ and $SVI(F,K)$ coincide.

Theorem 5. *If $SMVI(F,K)$ admits a strict solution, then, this is the unique solution of $SVI(F,K)$.*

Proof: Let $x^* \in K$ be a strict solution of $SMVI(F,K)$, that is:

$$\langle \xi, y - x^* \rangle > 0, \quad \forall y \in K, \; y \neq x^*, \quad \forall \xi \in F(y),$$

and assume, by contradiction, that there exists $x_1 \in K$, $x_1 \neq x^*$ such that x_1 solves $SVI(F,K)$. Hence we have:

$$\langle \xi, x_1 - x^* \rangle \leq 0, \quad \text{for some } \xi \in F(x_1),$$

which contradicts the fact that x^* solves $SMVI(F,K)$. □

A first link between variational inequalities and differential inclusions is given in the following proposition which has an immediate proof.

Proposition 2. *A point $x^* \in K$ is an equilibrium point for $DVI(F,K)$ if and only if it is a solution of $SVI(F,K)$.*

3 Variational Inequalities and Monotonicity of Trajectories

In this section we explore the links between variational inequalities and the stability of the trajectories of $DVI(F,K)$, w.r.t. function $\tilde{V}_{x^*}$.

Theorem 6. *If $x^* \in K$ is a solution of $SMVI(F,K)$, then every trajectory $x(t)$ of $DVI(F,K)$ is monotone w.r.t. function $\tilde{V}_{x^*}$.*

Proof: We observe that, under the hypotheses of the theorem, x^* is an equilibrium point of $DVI(F,K)$ (recall Lemma 1 and Proposition 2). Since $x(t)$ is differentiable a.e., so is $v(t) = \tilde{V}_{x^*}(x(t))$ and we have (at least a.e.):

$$\begin{aligned} v'(t) = \langle \tilde{V}'_{x^*}(x(t)), x'(t) \rangle = \langle x'(t), x(t) - x^* \rangle = \\ = \langle -\xi(x(t)) - n_K(x(t)), x(t) - x^* \rangle, \end{aligned}$$

where $\xi(x(t)) \in F(x(t))$ and $n_K(x(t)) \in N(K, x(t)))$. Hence $v'(t) \leq 0$ for a.a. $t \geq 0$ and hence, for $t_2 > t_1$:

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} v'(\tau) d\tau \leq 0.$$

□

Corollary 1. *Let x^* be an equlibrium point of $DVI(F, K)$ and assume that F is pseudomonotone. Then every trajectory of $DVI(F, K)$ is monotone w.r.t. function $\tilde{V}_{x^*}$.*

Proof: It is immediate combining Lemma 1 and Theorem 6 □

To get a sort of converse of the previous Theorem, we need the following result.

Theorem 7 ([1]). *Let K be a subset of $\mathbb{R}^n$ and let $V : K \to \mathbb{R}^+$ be a differentiable function. Assume that for all $x_0 \in K$, there exists $T > 0$ and a trajectory $x(\cdot)$ on $[0, T)$ of the differential inclusion $x'(t) = F(x(t))$, $x(0) = x_0$, satisfying:*

$$\forall s \geq t, \quad V(x(s)) - V(x(t)) \leq 0.$$

Then V is a Liapunov function for F, that is $\forall x \in K$, $\exists \xi \in F(x)$, such that $\langle V'(x), \xi \rangle \leq 0$.

Theorem 8. *Let x^* be an equilibrium point of $DVI(F, K)$. If for any point $x \in K$ there exists a trajectory of $DVI(F, K)$ starting at x and monotone w.r.t. function $\tilde{V}_{x^*}$, then x^* solves $WMVI(F, K)$.*

Proof: Let $\bar{x} \in \operatorname{ri} K$ (the relative interior of K) be the initial condition for a trajectory $x(t)$ of $DVI(F, K)$ and assume that $x(t)$ is monotone w.r.t. $\tilde{V}_{x^*}$. If we denote by L the smallest affine subspace generated by K and set $S = L - \bar{x}$, for $x \in K \cap U$, where U is a suitable neighborhood of $\bar{x}$, we have $T(K, x) = S$ and $N(K, x) = S^\perp$ (the subspace orthogonal to S). So, if $x(t)$ is a trajectory of $DVI(F, K)$ that starts at $\bar{x}$, then, for t "small enough", it remains in $\operatorname{ri} K \cap U$ and satisfies (recall Theorem 2):

$$\begin{cases} \text{for all } t \in [0, T], \ x(t) \in K \\ \text{for a.a. } t \in [0, T], \ x'(t) \in \pi_S(-F(x(t))) \end{cases}$$

Since S is a subspace, π_S is a linear operator; hence $\pi_S(-F(x(t))$ is compact and convex $\forall t \in [0, T]$ and furthermore $\pi_S(-F(\cdot))$ is u.s.c. .
Applying Theorem 7 we obtain the existence of a vector $\mu \in \pi_S(-F(\bar{x}))$, such that $\langle \tilde{V}'_{x^*}(\bar{x}), \mu \rangle \leq 0$. Taking into account inclusion (1), we have $\mu = -\xi(\bar{x}) - n(\bar{x})$, where $\xi(\bar{x}) \in F(\bar{x})$ and $n(\bar{x}) \in S^\perp$. Hence:

$$\begin{aligned} \langle \tilde{V}'_{x^*}(\bar{x}), \mu \rangle &= \langle -\xi(\bar{x}) - n(\bar{x}), \bar{x} - x^* \rangle = \\ &= \langle -\xi(\bar{x}), \bar{x} - x^* \rangle + \langle n(\bar{x}), x^* - \bar{x} \rangle \leq 0, \end{aligned}$$

from which it follows, since $\langle n(\bar{x}), x^* - \bar{x} \rangle = 0$:

$$\langle \xi(\bar{x}), \bar{x} - x^* \rangle \geq 0.$$

Since $\bar{x}$ is arbitrary in $\operatorname{ri} K$, we have:

$$\langle \xi(x), x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{ri} K.$$

Now, let $\tilde{x} \in \operatorname{cl} K \backslash \operatorname{ri} K$. Since $\operatorname{cl} K = \operatorname{cl} \operatorname{ri} K$, then $\tilde{x} = \lim x_k$, for some sequence $\{x_k\} \in \operatorname{ri} K$ and:

$$\langle \xi(x_k), x_k - x^* \rangle \geq 0, \forall k.$$

There exists a closed ball $\bar{B}(\tilde{x},\delta)$, with center in $\tilde{x}$ and radius δ, such that x_k is contained in the compact set $\bar{B}(\tilde{x},\delta)\cap K$ and since F is u.s.c., with compact images, the set:

$$\bigcup_{y\in\bar{B}(\tilde{x},\delta)\cap K} F(y)$$

is compact (see Proposition 3, p. 42 in [1]) and we can assume that $\xi(x_k)\to\tilde{\xi}\in\bigcup_{y\in\bar{B}(\tilde{x},\delta)\cap K}F(y)$. From the upper semicontinuity of F, it follows also $\tilde{\xi}\in F(\tilde{x})$ and so:

$$\langle\tilde{\xi},\tilde{x}-x^*\rangle\geq 0.$$

This completes the proof. □

Therorem 6 can be strenghtened with the following:

Theorem 9. *Let x^* be a strict solution of $SMVI(F,K)$, then:*

i) x^* is the unique equilibrium point of $DVI(F,K)$;

ii) every trajectory of $DVI(F,K)$, starting at a point $x_0\in K$ and defined on $[0,+\infty)$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.

Proof: The uniqueness of the equilibrium point follows from Theorem 5. The strict monotonicity of any trajectory $x(t)$ w.r.t. $\tilde{V}_{x^*}$ follows along the lines of the proof of Theorem 6. Now the proof of the convergence is an application of Liapunov function's technique.
Let $x(t)\in K$ be a solution of $DVI(F,K)$, starting at some point $x_0\in K$, i.e. with $x(0)=x_0$. Assume, ab absurdo, that $\alpha=\lim_{t\to+\infty}v(t)>0=\min_{y\in K}\tilde{V}_{x^*}(\cdot)$, where $v(t)=\tilde{V}_{x^*}(x(t))$. We observe that the limit defining α exists, because of the monotonicity of $v(\cdot)$ and to assume it differs from 0, it is equivalent to say that $x(t)\not\to x^*$. Thus, since $x(t)$ is monotone w.r.t. $\tilde{V}_{x^*}$, we have $\forall t\geq 0$:

$$\alpha\leq v(t)\leq\delta=\frac{\|x_0-x^*\|^2}{2}.$$

Letting $L:=\left\{x\in K\,:\,\alpha\leq\dfrac{\|x-x^*\|^2}{2}\leq\delta\right\}$, we have that L is a compact set and $x^*\notin L$, while $x(t)\in L$, $\forall t\geq 0$. Since x^* is a strict solution of $SMVI(F,K)$, we have:

$$\langle\xi,y-x^*\rangle<0,\quad\forall y\in K,\,y\neq x^*,\quad\forall\xi\in -F(y)$$

and, in particular:

$$\langle\xi,y-x^*\rangle<0,\quad\forall y\in L,\quad\forall\xi\in -F(y).$$

Now, we observe that there exists a number $m>0$, such that:

$$\max_{\xi\in -F(y)}\langle\xi,y-x^*\rangle\leq -m,\quad\forall y\in L.$$

Infact, if such a number does not exists, we would obtain the existence of sequences $y_n\in L$ and $\xi_n\in F(y_n)$, such that:

$$\langle\xi_n,y_n-x^*\rangle\geq-\frac{1}{n}.$$

Sending n to $+\infty$, we can assume that $y_n\to\bar{y}\in L$. Furthermore, since F is u.s.c. with compact images, the set:

$$\bigcup_{y\in L}F(y)$$

is compact and we can also assume $\xi_n \to \bar{\xi} \in \bigcup_{y \in L} F(y)$. By the upper semicontinuity of F, it follows also $\bar{\xi} \in F(\bar{y})$ and we get the absurdo:

$$\langle \bar{\xi}, \bar{y} - x^* \rangle \geq 0.$$

We have:

$$v'(t) = \langle x'(t), x(t) - x^* \rangle = \langle a(t) + b(t), x(t) - x^* \rangle,$$

with $a(t) \in -F(x(t))$, $b(t) \in -N(K, x(t))$ and hence:

$$v'(t) = \langle a(t), x(t) - x^* \rangle + \langle -b(t), x^* - x(t) \rangle.$$

Since $x(t) \in L$, for $t \geq 0$, we have $\langle a(t), x(t) - x^* \rangle \leq -m$, while $\langle -b(t), x^* - x(t) \rangle \leq 0$. Therefore $v'(t) \leq -m$, for $t \geq 0$. Now, we obtain, for $T > 0$:

$$v(T) - v(0) = \int_0^T v'(\tau) d\tau \leq -mT.$$

If $T = \dfrac{v(0)}{m}$, we get $v(T) \leq 0 = \min\limits_{y \in K} V(\cdot)$. But we also have:

$$v(T) \geq \alpha > \min_{y \in K} V(\cdot) = 0.$$

Hence a contradiction follows and we must have $\alpha = 0$, that is $x(t) \to x^*$.

□

Corollary 2. *Let x^* be an equlibrium point of $DVI(F, K)$ and assume that F is strictly pseudomonotone. Then properties i) and ii) of the previous Theorem hold.*

Proof: It is immediate combining Lemma 1 and Theorem 9. □

Example 1. *Let $K = \mathbb{R}^2$ and consider the system of autonomous differential equations:*

$$x'(t) = -F(x(t)),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a single-valued map defined as:

$$F(x, y) = \begin{bmatrix} -y + x|1 - x^2 - y^2| \\ x + y|1 - x^2 - y^2| \end{bmatrix}$$

Clearly $(x^*, y^*) = (0, 0)$ is an equilibrium point and one has $\langle F(x, y), (x, y) \rangle \geq 0 \ \forall (x, y) \in \mathbb{R}^2$, so that $(0, 0)$ is a solution of $SMVI(F, K)$ and hence, according to Theorem 6, every solution $x(t)$ of the considered system of differential equations is monotone w.r.t. $\tilde{V}_{x^*}$. Anyway, not all the solution of the system converge to $(0, 0)$. Infact, passing to polar coordinates, the system can be written as:

$$\begin{cases} \rho'(t) = -\rho(t)|1 - \rho^2(t)| \\ \theta'(t) = -1 \end{cases}$$

and solving the system, one can easily see that the solutions that start at a point (ρ, θ), with $\rho \geq 1$ do not converge to $(0, 0)$, while the solutions that start at a point (ρ, θ) with $\rho < 1$ converge to $(0, 0)$. This last fact could be checked observing that for every $c < 1$, $(0, 0)$ is a strict solution of $SMVI(F, K_c)$ where:

$$K_c := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < c\}.$$

4 The case in which F is single-valued and K is open

In this section we focus on the case in which F is a single valued operator from K to $\mathbb{R}^n$ and K is open (and convex). In this case $N(K,x) = \{0\}, \quad \forall x \in K$ and $DVI(F,K)$ reduces to the classical autonomous system:

$$DS(F) \quad x'(t) = -F(x(t)).$$

Clearly, now $x^* \in K$ is an equilibrium point of $DS(F)$ when $F(x^*) = 0$. In [18] the authors give some necessary and sufficient conditions for the existence of monotone trajectories of $DS(F)$ under the hypothesis that F is of class C^1. Anyway, existence and uniqueness of the solutions of Problem 1 hold under weaker hypotheses. In particular, we recall the following classical result (see e.g. [12]).

Theorem 10. *Let K be an open subset of $\mathbb{R}^n$, $x_0 \in K$ and let F be Lipschitz with constant K on a neighborhood U of x_0 with radius δ, with $\max_{x\in U} |F(x)| \leq M$. If $0 < a < \min\{\delta/M, 1/K\}$, then there is a unique (differentiable) function $x : [0,a) \to K$, such that $x(0) = x_0$ and $x'(t) = -F(x(t)), \quad \forall t \in [0,a)$.*

Here we generalize the results in [18] to the case in which F is locally Lipschitz. We will give necessary and sufficient conditions for the existence of monotone trajectories of $DS(F)$ (w.r.t. function $\tilde{V}_{x^*}$), expressed by means of Clarke's generalized Jacobian. We remember the following definition (see for instance [6]):

Definition 9. *Let G be a locally Lipschitz function from K to $\mathbb{R}^m$. Clarke's generalized Jacobian of G at x is the subset of the space $\mathbb{R}^{n\times m}$ of $n \times m$ matrices, defined as:*

$$J_C G(x) = \mathrm{conv}\{\lim JG(x_k) : x_k \to x,\ G \text{ is differentiable at } x_k\}$$

(here JG denotes the Jacobian of G and $\mathrm{conv}A$ stands for the convex hull of the set $A \subseteq \mathbb{R}^n$).

The following proposition summarizes the main properties of the generalized Jacobian.

Proposition 3.

i) $J_C F(x)$ is a nonempty, convex and compact subset of $\mathbb{R}^{n\times m}$;

ii) the map $x \to J_C F(x)$ is u.s.c.;

iii) (Mean value Theorem) For all $x, y \in K$ we have $F(y)-F(x) \in \mathrm{conv}\{J_C F(x+\delta(y-x))(y-x), \delta \in [0,1]\}$.

Definition 10. *Let $\mathcal{A}(\cdot)$ be a map from $\mathbb{R}^n$ into the subsets of the space $\mathbb{R}^{n\times n}$ of $n\times n$ matrices. We say that $\mathcal{A}(\cdot)$ is positively defined at x (respectively weakly positively defined) when:*

$$\inf_{A\in\mathcal{A}(x)} u^\top A u \geq 0, \quad \forall u \in \mathbb{R}^n$$
$$\left(\ \sup_{A\in\mathcal{A}(x)} u^\top A u \geq 0, \quad \forall u \in \mathbb{R}^n \ \right).$$

If the inequality is strict (for $u \neq 0$), we say that $\mathcal{A}(x)$ is strictly positive defined (resp. strictly weakly positive defined).

Theorem 11. *Let $F : K \to \mathbb{R}^n$ be locally Lipschitz and let x^* be an equilibrium point of $DS(F)$. If there exists a positive number δ such that for any $x_0 \in K$ with $\|x_0 - x^*\| < \delta$, there exists a trajectory $x(t)$ of $DS(F)$ starting at x_0 and monotone w.r.t. $\tilde{V}_{x^*}$, then Clarke's generalized Jacobian of F at x^* is weakly positively defined.*

Proof: Let $B(x^*, \delta)$ be the open ball with center in x^* and radius δ. Fix $z \in B(x^*, \delta)$ and let $y(\alpha) = x^* + \alpha(z - x^*)$, for $\alpha \in [0, 1]$ (clearly $y(\alpha) \in B(x^*, \delta)$). Let $x(t)$ be a trajectory of $DS(F)$ starting at $y(\alpha)$; for $v(t) = \tilde{V}_{x^*}(x(t))$, we have:

$$0 \geq v'(0) = \langle x'(0), y(\alpha) - x^* \rangle,$$

and:

$$x'(0) = -F(y(\alpha)),$$

so that:

$$\langle F(y(\alpha), y(\alpha) - x^* \rangle \geq 0.$$

Now we have, applying the mean value theorem:

$$F(y(\alpha)) - F(x^*) = F(y(\alpha)) \in \mathrm{conv}\big\{\alpha J_C F(x^* + \delta(z - x^*))(z - x^*),\, \delta \in [0, \alpha]\big\} = A(\alpha).$$

Since $J_C F(\cdot)$ is u.s.c., $\forall \varepsilon > 0$ and for δ "small enough", let's say $\delta \in [0, \beta(\epsilon)]$ we have:

$$J_C F(x^* + \delta(z - x^*)) \subseteq J_C F(x^*) + \varepsilon B := J_\varepsilon F(x^*)$$

(here B denotes the open unit ball in $\mathbb{R}^{n \times n}$). So, it follows, for $\alpha = \beta(\varepsilon)$:

$$A(\beta(\varepsilon)) \subseteq \beta(\varepsilon) J_\varepsilon F(x^*)(z - x^*),$$

and hence, for any $\varepsilon > 0$, $F(y(\beta(\varepsilon))) \in \beta(\varepsilon) J_\varepsilon F(x^*)(z - x^*)$.
Now, let $\varepsilon_n = 1/n$ and $\alpha_n = \beta(\varepsilon_n)$. We have $\langle F(y(\alpha_n)), y(\alpha_n) - x^* \rangle \geq 0$, that is:

$$\alpha_n^2 (z - x^*)^\top (d(\alpha_n) + \gamma(\alpha_n))(z - x^*) \geq 0,$$

with $\gamma(\alpha_n) \in \frac{1}{n} B$ and $d(\alpha_n) \in J_C F(x^*)$. So we obtain:

$$(z - x^*)^\top d(\alpha_n)(z - x^*) \geq -(z - x^*)^\top \gamma(\alpha_n)(z - x^*) = -\frac{1}{n}(z - x^*) b_n (z - x^*),$$

with $b_n \in B$. Sending n to $+\infty$ we can can assume $d(\alpha_n) \to d \in J_C F(x^*)$ and we get:

$$(z - x^*)^\top d\,(z - x^*) \geq 0.$$

Since z is arbitary in $B(x^*, \delta)$, we obtain that $J_C F(x^*)$ is weakly positive defined.

□

Example 2. *The condition of the previous Theorem is necessary but not sufficient for the existence of monotone trajectories (w.r.t. $\tilde{V}$). Consider the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ defined as:*

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and the autonomous differential equation $x'(t) = -F(x(t))$. Clearly $x^* = 0$ is an equilibrium point and it is known that $J_C F(0) = [-1, 1]$. Hence the necessary condition of Theorem 11 is satisfied, but it is easily seen that any trajectory $x(t)$ of the considered differential equation (apart from the trivial solution $x(t) \equiv 0$) is not monotone w.r.t. $\tilde{V}_{x^*}$.

Theorem 12. *Assume that $J_C F(x^*)$ is strictly positively defined. Then, every trajectory $x(t)$ of $DS(F)$ starting "sufficiently near" x^* and defined on $[0, +\infty)$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.*

Proof: By assumption:

$$\inf_{A\in J_C F(x^*)} u^\top A u > 0, \quad \forall u \in \mathbb{R}^n \backslash \{0\},$$

and this condition is equivalent to the existence of a positive number m such that $\inf_{A\in J_C F(x^*)} v^\top A v > m, \;\; \forall v \in S^1$ (the unit sphere in $\mathbb{R}^n$). Let $\varepsilon > 0$ and consider the set:

$$J_\varepsilon F(x^*) := J_C F(x^*) + \varepsilon B.$$

We claim:

$$\inf_{A\in J_\varepsilon F(x^*)} u^\top A u > 0, \quad \forall u \in \mathbb{R}^n \backslash \{0\},$$

for ε "small enough". Indeed, $A \in J_\varepsilon F(x^*)$ if and only if $A = A' + A''$, with $A' \in J_C F(x^*)$ and $A'' \in \varepsilon B$ and hence, for $u \in \mathbb{R}^n\backslash\{0\}$:

$$\inf_{A\in J_\varepsilon F(x^*)} u^\top A u \geq \inf_{A'\in J_C F(x^*)} u^\top A' u + \inf_{A''\in \varepsilon B} u^\top A'' u.$$

Since $A'' \in \varepsilon B$, we have $|u^\top A'' u| \leq \|A''\| \|u\| \leq \varepsilon \|u\|^2$ and we get:

$$\inf_{A'\in J_C F(x^*)} u^\top A' u + \inf_{A''\in \varepsilon B} u^\top A'' u \geq \inf_{A'\in J_C F(x^*)} u^\top A' u - \varepsilon \|u\|^2.$$

Therefore:

$$\inf_{A\in J_\varepsilon F(x^*)} \frac{u^\top A u}{\|u\|^2} \geq \inf_{A'\in J_C F(x^*)} \frac{u^\top A' u}{\|u\|^2} - \varepsilon$$

and for $\varepsilon < m$, the righthandside is positive.
If we fix ε in $(0, m)$, for a suitable $\delta > 0$ we have, for all $x \in \bar{B}(x^*, \delta)$:

$$J_C F(x^* + \alpha(x - x^*)) \subseteq J_\varepsilon F(x^*), \quad \forall \alpha \in (0, 1)$$

and from the mean value theorem, we obtain:

$$F(x) = F(x) - F(x^*) \in \operatorname{conv}\left\{ J_C F(x^* + \delta(x - x^*))(x - x^*), \; \delta \in [0, 1] \right\} \subseteq J_\varepsilon F(x^*)(x - x^*).$$

Hence we conclude:

$$\langle F(x), x - x^* \rangle > 0, \quad \forall x \in (\mathbb{R}^n \cap \bar{B}(x^*, \delta)) \backslash \{x^*\}.$$

and so x^* is a strict solution of $SMVI(F, \mathbb{R}^n \cap \bar{B}(x^*, \delta))$. The proof now follows from Theorem 9.

□

Example 3. *The condition of the previous Theorem is sufficient bur not necessary for the monotonicity of trajectories. Consider the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ defined as:*

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x} + ax, & x \neq 0 \\ 0, \quad x = 0 \end{cases}$$

where $0 < a < 1$, and the autonomous differential equation $x'(t) = -F(x(t))$, for which $x^* = 0$ is an equilibrium point. In a suitable closed neighborhood U of 0 we have $F(x) > 0$ if $x > 0$, while $F(x) < 0$, if $x < 0$ and hence x^* is a strict solution of $SMVI(F, U)$. It follows that every solution of the considered differential equation, starting "near" x^*, is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to 0. If we calculate the generalized Jacobian of F at 0 we get $J_C F(0) = [-1+a, 1+a]$ and the sufficient condition of the previous Theorem is not satisfied.

5 An application: generalized gradient inclusions

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on the open set Ω. Equations of the form:

$$x'(t) = -f'(x(t)), \quad x(0) = x_0$$

are called "gradient equations" (see for instance [12]). In [1] an extesion of the classical gradient equation to the case in which f is a lower semicontinuous convex function is considered, replacing the above gradient equation, with the differential inclusion:

$$x'(t) = -\partial f(x(t)), \quad x(0) = x_0,$$

where ∂f denotes the subgradient of f. In this section K will denote again a closed convex subset of $\mathbb{R}^n$

Here, we consider a locally Lipschitz function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, where Ω is an open set containing the closed convex set K, and the DVI:

$$DVI(\partial_C f, K) \qquad \begin{cases} \forall t \in [0,T], \quad x(t) \in K, \\ \text{for a.a. } t \in [0,T], \quad x'(t) \in -\partial_C f(x(t)) - N(K, x(t)) \end{cases}$$

where $\partial_C f(x)$ denotes Clarke's generalized gardient of f at x, with the aim of studying the behaviour of its trajectories. (The definition of Clarke's generalized gradient can be recovered from Definition 9, putting there $m = 1$).

Definition 11. *[15] We say that $\partial_C f$ is semistrictly pseudomonotone on K, when for every $x, y \in K$, with $f(x) \neq f(y)$, we have:*

$$\exists u \in \partial_C f(x) : \ \langle u, y - x \rangle \geq 0 \Rightarrow \forall v \in \partial_C f(y) : \ \langle v, y - x \rangle > 0.$$

Clearly, if $\partial_C f$ is strictly pseudomonotone, then it is also semistrictly pseudomonotone.

Definition 12. *i) f is said to be pseudoconvex on K when $\forall x, y \in K$, with $f(y) > f(x)$, there exixts a positive number $a(x,y)$, depending on x and y and a number $\delta(x,y) \in (0,1]$, such that:*

$$f(\lambda x + (1-\lambda)y) \leq f(y) - \lambda a(x,y), \quad \forall \lambda \in (0, \delta(x,y)).$$

ii) f is said to be strictly pseudoconvex if the previous inequality holds whenever $f(y) \geq f(x), \quad x \neq y$.

We recall the following result, obtained by Luc [15].

Theorem 13. *i) Assume that $\partial_C f$ is semistrictly pseudomonotone on an open convex set $A \subseteq \mathbb{R}^n$. Then f is pseudoconvex on A.*

ii) Assume that $\partial_C f$ is strictly pseudomonotone on an open convex set A. Then f is strictly pseudoconvex on A.

Remark 5. Strictly pseudomonotone and semistrictly pseudomonotone maps are called respectively "strictly quasimonotone" and "semistrictly quasimonotone" in [15].

Definition 13. *We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is inf-compact on the closed convex set K, when $\forall c \in \mathbb{R}$, the level sets:*

$$lev_{\leq c} f := \{x \in K : f(x) \leq c\}$$

are compact.

Remark 6. Clearly, if f is inf-compact on K the set $\operatorname{argmin}(f,K)$ of minimizers of f over K is compact. The converse does not hold.

Theorem 14. *Let $x(t)$ be a slow solution of $DVI(\partial_C f, K)$ defined on $[0,T]$. Then, $\forall s_1,\ s_2 \in [0,T]$ with $s_2 \geq s_1$, we have:*

$$f(x(s_2)) - f(x(s_1)) \leq -\int_{s_1}^{s_2} \|m(-\partial_C f(x(s)) - N(K,x(s)))\|^2 ds.$$

Hence the function $g(t) = f(x(t))$ is nonincreasing and $\lim_{t\to+\infty} f(x(t))$ exists.

Proof: Since a locally Lipschitz function is differentiable a.e., the function $g(t) = f(x(t))$ is differentiable a.e., with $g'(t) = f'(x(t))x'(t)$ and $x'(t) \in m(-\partial_C f(x(t)) - N(K,x(t)))$ for a.a. t. Recalling (Theorem 3) that the slow solutions of $DVI(\partial_C f, K)$ coincide with the slow solutions of $PDI(\partial_C f, K)$ and that $f'(x(t)) \in \partial_C f(x(t))$ [6], we have from Proposition 1:

$$\sup_{z\in\partial_C f(x(t))} \langle z, m(-\partial_C f(x(t)) - N(K,x(t)))\rangle + \|m(-\partial_C f(x(t)) - N(K,x(t)))\|^2 \leq 0$$

and for a.a. t, we get:

$$g'(t) = f'(x(t))x'(t) \leq -\|m(-\partial_C f(x(t)) - N(K,x(t)))\|^2 \leq 0,$$

from which we deduce:

$$f(x(s_2)) - f(x(s_1)) \leq -\int_{s_1}^{s_2} \|m(-\partial_C f(x(s)) - N(K,x(s)))\|^2 ds \leq 0.$$

The second part of the theorem is now an immediate consequence. $\square$

Theorem 15. *Suppose that f achieves its minimum over K at some point. Assume that $\partial_C f$ is a semistrictly pseudomonotone map and that f is inf-compact. Then every slow solution $x(t)$ of $DVI(\partial_C f, K)$ defined on $[0,+\infty)$, is such that:*

$$\lim_{t\to+\infty} f(x(t)) = \min_{x\in K} f(x).$$

Furthermore, every cluster point of $x(t)$ is a minimum point for f over K.

Proof: Let $x(t)$ be a slow solution starting at $x_0 = x(0)$ and ab absurdo, assume that $\lim_{t\to+\infty} f(x(t)) = \alpha > \min_{x\in K} f(x)$. The set:

$$Z = \{x \in K : \alpha \leq f(x) \leq f(x_0)\}.$$

is compact, since f is inf-compact and $\operatorname{argmin}(f,K) \cap Z = \emptyset$. If we set $A = \{x(t), t \in [0,+\infty)\}$, then we get $\operatorname{cl} A \subseteq Z$, and hence $\operatorname{argmin}(F,K) \cap \operatorname{cl} A = \emptyset$. If $x^* \in \operatorname{argmin}(f,K)$, then it is an equilibrium point of $DVI(\partial_C f, K)$ (see [6]), that is:

$$0 \in \partial_C f(x^*) + N(K,x^*),$$

and this is equivalent (see Proposition 2) to the fact that x^* solves $SVI(\partial_C f, K)$, that is to the existence of vector $v \in \partial_C f(x^*)$ such that:

$$\langle v, x - x^*\rangle \geq 0, \quad \forall x \in K.$$

It follows also: $\langle v, a - x^* \rangle \geq 0, \quad \forall a \in \mathrm{cl}\, A$ and since $\partial_C f$ is semistrictly quasimonotone, we have (observe that $f(a) \neq f(x^*)$ $\forall a \in \mathrm{cl}\, A$):

$$\langle w, a - x^* \rangle < 0, \quad \forall w \in -\partial_C f(a), \quad \forall a \in \mathrm{cl}\, A.$$

Observing that $\mathrm{cl}\, A$ is a compact set, as in the proof of Theorem 9, it follows the existence of a positive number m such that:

$$\langle w, a - x^* \rangle < -m, \quad \forall w \in -\partial_C f(a), \quad \forall a \in \mathrm{cl}\, A.$$

Hence, letting $v(t) = \frac{\|x(t)-x^*\|^2}{2}$, as in the proof of Theorem 9, we obtain $v'(t) \leq -m$ for a.a. t and hence, for $T > 0$:

$$v(T) - v(0) = \int_0^T v'(\tau) d\tau \leq -mT.$$

For $T = v(0)/m$, we obtain $v(T) \leq 0$, that is $v(T) = 0$ and hence $x(T) = x^*$, but this is absurdo, since the set A does not intersects $\mathrm{argmin}(f, K)$.
Now the last assertion of the theorem is obvious.

□

The previuos Theorem can be strenghtened using the results of section 3.

Proposition 4. *Let f be a function that achieves its minimum over K at some point x^* and assume that x^* is a strict solution of $SMVI(\partial_C f, K)$. Then every solution defined on $[0, +\infty)$ of $DVI(\partial_C f, K)$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.*

Proof: It is immediate recalling that if x^* is a minimum point for f over K, then it is an equilibrium point of $DVI(\partial_C f, K)$ and applying Theorem 9. □

Remark 7. If x^* is a strict solution of $SMVI(\partial_C f, K)$, then it can be proved that f is strictly increasing along rays starting at x^*. The proof is similar to that of Proposition 4 in [7].

Corollary 3. *Let f be a function that achieves its minimum over K at some point x^*. If $\partial_C f$ is strictly pseudomonotone, then x^* is the unique minimum point for f over K and every solution of $DVI(\partial_C f, K)$ defined on $[0, +\infty)$ converges to x^*.*

Proof: Recall that, under the hypotheses, f is strictly pseudoconvex (Theorem 13) and hence it follows easily that x^* is the unique minimum point of f over K. The proof is now an immediate consequence of Corollary 2. □

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