THE GENERALIZATION ON A SPECIAL CASE OF OPPENHIM THEOREM

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ABSTRACT. In this paper, we give the following generalization on a special case of Oppenhim theorem: Let $1 \leqslant m$, and $a_2 = (b_1^m + c_1^m)^{\frac{1}{m}}, b_2 = (c_1^m + a_1^m)^{\frac{1}{m}}, c_2 = (a_1^m + b_1^m)^{\frac{1}{m}}$, then the inequalities hold $p_2 \geqslant 2^{\frac{1}{m}} p_1$, and $\Delta_2 \geqslant 2^{\frac{2}{m}} \Delta_1$.

1. Introduction and Notation

Throughout the paper we assume p_1 denote the semi-perimeter of triangle $A_1B_1C_1$, a_1, b_1, c_1 the opposite sides, R_1 the circumradius, r_1 the inradius and $\triangle_1$ the area. Similarly, one defines triangle $A_2B_2C_2$ and triangle $A_3B_3C_3$.

In1963, A.Oppenhim [1] obtained the following theorem:

Theorem 1.1. *Let $1 \leqslant m \leqslant 4$, and*

$$a_3 = (a_1^m + a_2^m)^{\frac{1}{m}}, b_3 = (b_1^m + b_2^m)^{\frac{1}{m}}, c_3 = (c_1^m + c_2^m)^{\frac{1}{m}}, \tag{1.1}$$

then the following inequalities hold

$$p_3 \geqslant 2^{\frac{1}{m}-1}(p_1 + p_2), \tag{1.2}$$

and

$$\Delta_3 \geqslant 2^{\frac{2}{m}-1}(\Delta_1 + \Delta_2). \tag{1.3}$$

When $m > 4$, a negation of the inequalities (1.2)and (1.3) was obtained in [2].

In this paper, we give a generalization for a special case of Theroem1.1.

2. Main Result and Lemma

Theorem 2.1. *If m be a real number for $1 \leqslant m$, and*

$$a_2 = (b_1^m + c_1^m)^{\frac{1}{m}}, b_2 = (c_1^m + a_1^m)^{\frac{1}{m}}, c_2 = (a_1^m + b_1^m)^{\frac{1}{m}} \tag{2.1}$$

then the following inequalities hold

$$p_2 \geqslant 2^{\frac{1}{m}} p_1 \tag{2.2}$$

and

$$\Delta_2 \geqslant 2^{\frac{2}{m}} \Delta_1. \tag{2.3}$$

To prove Theorem2.1, we will use the following lemma:

Lemma 2.1. *If $0 < u \leqslant 1, v > 0$, and $\lambda < 1$, then*

$$(v+1)^\lambda - (v+u)^\lambda \leqslant (4v)^{\frac{\lambda-1}{2}}(1 - u^{\frac{\lambda+1}{2}}) \tag{2.4}$$

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Proof. Set a function

$$f(u) = (u+v)^\lambda + (4v)^{\frac{\lambda-1}{2}}(1-u^{\frac{\lambda+1}{2}}) - (1+v)^\lambda, (0 < u \leqslant 1, v > 0, \lambda < 1)$$

We have

$$\frac{\partial f(u)}{\partial u} = \lambda(u+v)^{\lambda-1} - (\frac{\lambda+1}{2})(4uv)^{\frac{\lambda-1}{2}}$$

By using the arithmetic-geometric mean inequality, we obtain

$$\frac{\partial f(u)}{\partial u} \leqslant \lambda(2\sqrt{uv})^{\lambda-1} - (\frac{\lambda+1}{2})(4uv)^{\frac{\lambda-1}{2}} = (\frac{\lambda-1}{2})(4uv)^{\frac{\lambda-1}{2}} < 0,$$

since

$$\lambda - 1 < 0, u > 0, v > 0.$$

Therefore, the function f with fixation v ($v > 0$) be monotonically decreasing in $u \in (0, 1]$, that is

$$f(u) \geqslant f(1) = 0$$

i.e., the inequality (2.4) is true. This completes the proof. ∎

3. The Proof of Theroem2.1

Proof. From $a_2, b_2, c_2 > 0, m - 1 \geqslant 0$, we have

$$\begin{aligned}(a_2+b_2)^m &= (a_2+b_2)(a_2+b_2)^{m-1} = a_2(a_2+b_2)^{m-1} + b_2(a_2+b_2)^{m-1} \\ &> a_2(a_2)^{m-1} + b_2(b_2)^{m-1} = (a_2)^m + (b_2)^m \\ &= b_1^m + c_1^m + c_1^m + a_1^m > a_1^m + b_1^m = c_2^m\end{aligned}$$

therefore $a_2 + b_2 > c_2$. Similarly, we obtain $b_2 + c_2 > a_2$ and $c_2 + a_2 > b_2$. That is to say, a_2, b_2, c_2 are three sides of the triangle $\Delta A_2B_2C_2$.

By using the power mean inequality, we have

$$\begin{aligned}p_2 &= \frac{1}{2}[(a_1^m + b_1^m)^{\frac{1}{m}} + (b_1^m + c_1^m)^{\frac{1}{m}} + (c_1^m + a_1^m)^{\frac{1}{m}}] \\ &\geqslant 2^{\frac{1}{m}-2}(a_1+b_1) + 2^{\frac{1}{m}-2}(b_1+c_1) + 2^{\frac{1}{m}-2}(c_1+a_1) = 2^{\frac{1}{m}}p_1\end{aligned}$$

where $m \geqslant 1$. This completes the proof of inequality (2.2).

In order to prove inequality(2.3), We first prove the following inequality:

$$(a_1^m + b_1^m)^{\frac{2}{m}} + (b_1^m + c_1^m)^{\frac{2}{m}} - (c_1^m + a_1^m)^{\frac{2}{m}} \leqslant 2^{\frac{2-m}{m}}(b_1^2 + a_1b_1 + b_1c_1 - c_1a_1) \tag{3.1}$$

Let $\frac{a_1}{b_1} = x^{\frac{1}{m}}$ $(x \geqslant 1)$, and $\frac{c_1}{b_1} = t^{\frac{1}{m}}$ $(0 < t \leqslant 1)$, then

$$\begin{aligned}&b_1^{-2}[(a_1^m + b_1^m)^{\frac{2}{m}} + (b_1^m + c_1^m)^{\frac{2}{m}} - (c_1^m + a_1^m)^{\frac{2}{m}} - 2^{\frac{2-m}{m}}(b_1^2 + a_1b_1 + b_1c_1 - c_1a_1)] \\ &= (x+1)^{\frac{2}{m}} + (t+1)^{\frac{2}{m}} - (x+t)^{\frac{2}{m}} - 2^{\frac{2-m}{m}}[x^{\frac{1}{m}}(1-t^{\frac{1}{m}}) + t^{\frac{1}{m}} + 1]\end{aligned}$$

Set

$$g(x) = (x+1)^{\frac{2}{m}} + (t+1)^{\frac{2}{m}} - (x+t)^{\frac{2}{m}} - 2^{\frac{2-m}{m}}[x^{\frac{1}{m}}(1-t^{\frac{1}{m}}) + t^{\frac{1}{m}} + 1]$$

we have

$$\frac{\partial g(x)}{\partial x} = \frac{2}{m}[(x+1)^{\frac{2-m}{m}} - (x+t)^{\frac{2-m}{m}} - (4x)^{\frac{1-m}{m}}(1-t^{\frac{1}{m}})]$$

From Lemma2.1 and $0 < t \leqslant 1, x \geqslant 1, \dfrac{2-m}{m} < 1$, we obtain $\dfrac{\partial g(x)}{\partial x} < 0$. Therefore, the function g with fixation t $(0 < t \leqslant 1)$ be monotonically decreasing in $x \in [1, +\infty)$, that is $g(x) \leqslant g(1) = 0$. i.e., the inequality (3.1) is true.

Secondly, to proved inequality (2.3).

By using the power mean inequality, we have

$$(a_1^m + b_1^m)^{\frac{2}{m}}(b_1^m + c_1^m)^{\frac{2}{m}} \geqslant 2^{\frac{4-4m}{m}}(a_1+b_1)^2(b_1+c_1)^2 \quad (m \geqslant 1)$$

From Heron's formula [3] and (3.1), we follow

$$\begin{aligned}\Delta_2^2 &= \frac{1}{4}[c_2^2a_2^2 - \frac{1}{4}(c_2^2+a_2^2-b_2^2)^2]\\ &= \frac{1}{4}(a_1^m+b_1^m)^{\frac{2}{m}}(b_1^m+c_1^m)^{\frac{2}{m}} - \frac{1}{16}[(a_1^m+b_1^m)^{\frac{2}{m}}+(b_1^m+c_1^m)^{\frac{2}{m}}-(c_1^m+a_1^m)^{\frac{2}{m}}]^2\\ &\geqslant \frac{1}{4}(a_1^m+b_1^m)^{\frac{2}{m}}(b_1^m+c_1^m)^{\frac{2}{m}} - \frac{1}{16}[2^{\frac{2-m}{m}}(b_1^2+a_1b_1+b_1c_1-c_1a_1)]^2\\ &\geqslant \frac{1}{4}[2^{\frac{4-4m}{m}}(a_1+b_1)^2(b_1+c_1)^2] - \frac{1}{16}[2^{\frac{2-m}{m}}(b_1^2+a_1b_1+b_1c_1-c_1a_1)]^2\\ &= 2^{\frac{4-6m}{m}}[(a_1+b_1)^2(b_1+c_1)^2-(b_1^2+a_1b_1+b_1c_1-c_1a_1)^2]\\ &= 2^{\frac{4-4m}{m}}a_1b_1c_1(a_1+b_1+c_1)\end{aligned}$$

According to the expansion $a_1b_1c_1 = 4R_1\Delta_1$, $a_1+b_1+c_1 = \frac{2\Delta_1}{r_1}$ for triangle, and Euler's inequality $R_1 \geqslant 2r_1$, we have

$$\Delta_2^2 \geqslant 2^{\frac{4-4m}{m}}a_1b_1c_1(a_1+b_1+c_1) = 2^{\frac{4}{m}}\Delta_1{}^2(\frac{R_1}{2r_1}) \geqslant (2^{\frac{2}{m}}\Delta_1)^2.$$

Rearranging we obtain inequality (2.3).This completes the proof . ∎

4. The Analogue of Oppenhim's Inequality and an Open Problem

Theorem 4.1. *Let* $a_2 = (b_1^2+c_1^2)^{\frac{1}{2}}, b_2 = (c_1^2+a_1^2)^{\frac{1}{2}}, c_2 = (a_1^2+b_1^2)^{\frac{1}{2}}$, *then the following inequality holds*

$$r_2 \geqslant \sqrt{2}\, r_1 \tag{4.1}$$

Proof. From Heron's formula, we have

$$\begin{aligned}\Delta_2{}^2 &= \frac{1}{4}[c_2^2a_2^2 - \frac{1}{4}(c_2^2+a_2^2-b_2^2)^2]\\ &= \frac{1}{4}(a_1^2+b_1^2)(b_1^2+c_1^2) - \frac{1}{16}[(a_1^2+b_1^2)+(b_1^2+c_1^2)-(c_1^2+a_1^2)]^2\\ &= \frac{1}{4}(a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2),\end{aligned}$$

or

$$\Delta_2 = \frac{1}{2}\sqrt{a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2},$$

and

$$r_2 = \frac{\Delta_2}{p_2} = \frac{\sqrt{a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2}}{\sqrt{a_1^2+b_1^2}+\sqrt{b_1^2+c_1^2}+\sqrt{c_1^2+a_1^2}},$$

therefore inequality(4.1) is equivalent to

$$\frac{a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2}{(\sqrt{a_1^2+b_1^2}+\sqrt{b_1^2+c_1^2}+\sqrt{c_1^2+a_1^2})^2} \geqslant 2r_1^2. \tag{4.2}$$

Utilizing the fact that

$$(\sqrt{a_1^2+b_1^2}+\sqrt{b_1^2+c_1^2}+\sqrt{c_1^2+a_1^2})^2 \leqslant 6(a_1^2+b_1^2+c_1^2),$$

$$a_1^2+b_1^2+c_1^2 = 2(p_1^2-4R_1r_1-r_1^2),$$

and

$$a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2 = (p_1^2-4R_1r_1-r_1^2)^2+4p_1^2r_1^2,$$

therefore

$$\begin{aligned}
&\frac{a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2}{(\sqrt{a_1^2+b_1^2}+\sqrt{b_1^2+c_1^2}+\sqrt{c_1^2+a_1^2})^2}\\
&\geqslant \frac{a_1^2b_1^2+b_1^2c_1^2+c_1^2a_1^2}{6(a_1^2+b_1^2+c_1^2)}\\
&=\frac{1}{12}(p_1^2-4R_1r_1-r_1^2+\frac{4p_1^2r_1^2}{p_1^2-4R_1r_1-r_1^2}).
\end{aligned}$$

By using the arithmetic-geometric mean inequality and well known inequality $p_1 \geqslant 3\sqrt{3}r_1$, we obtain

$$\frac{1}{3}(p_1^2-4R_1r_1-r_1^2)+\frac{4p_1^2r_1^2}{p_1^2-4R_1r_1-r_1^2}\geqslant \frac{4}{\sqrt{3}}p_1r_1\geqslant 12r_1^2. \tag{4.3}$$

From Gerretsen's inequality (see[4]) $p_1^2 \geqslant 16R_1r_1 - 5r_1^2$ and Euler's inequality $R_1 \geqslant 2r_1$, we get

$$\frac{2}{3}(p_1^2-4R_1r_1-r_1^2)\geqslant \frac{2}{3}(12R_1r_1-6r_1^2)\geqslant 12r_1^2. \tag{4.4}$$

Combining inequalities (4.3) and (4.4), we have

$$\frac{1}{12}(p_1^2-4R_1r_1-r_1^2+\frac{4p_1^2r_1^2}{p_1^2-4R_1r_1-r_1^2})\geqslant 2r_1^2, \tag{4.5}$$

also inequality (4.2) holds. The proof of Theorem4.1 is completed. ∎

Finally, we propose an open problem:

Prove that: If m be a real number for $m \geqslant 1$, and $a_2 = (b_1^m + c_1^m)^{\frac{1}{m}}, b_2 = (c_1^m + a_1^m)^{\frac{1}{m}}, c_2 = (a_1^m + b_1^m)^{\frac{1}{m}}$, then the following inequality holds

$$r_2 \geqslant 2^{\frac{1}{m}}r_1. \tag{4.6}$$

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