REVERSES OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY FOR $n-$TUPLES OF COMPLEX NUMBERS

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Abstract. Some new reverses of the Cauchy-Bunyakovsky-Schwarz inequality for $n-$tuples of real and complex numbers related to Cassels and Shisha-Mond results are given.

1. Introduction

Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be two positive n-tuples with the property that there exists the positive numbers m_i, M_i $(i = 1, 2)$ such that

$$0 < m_1 \leq a_i \leq M_1 < \infty \quad \text{and} \quad 0 < m_2 \leq b_i \leq M_2 < \infty, \tag{1.1}$$

for each $i \in \{1, \dots, n\}$.

The following reverses of the Cauchy-Bunyakovsky-Schwarz (CBS) inequality are well known in the literature:

(1) **Pólya-Szegö's inequality** [8]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2; \tag{1.2}$$

(2) **Shisha-Mond's inequality** [9]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leq \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}} \right)^2; \tag{1.3}$$

(3) **Ozeki's inequality** [7]

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k \right)^2 \leq \frac{1}{4} n^2 \left(M_1 M_2 - m_1 m_2 \right)^2; \tag{1.4}$$

(4) **Diaz-Metcalf's inequality** [1]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^n a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^n a_k b_k. \tag{1.5}$$

If the weight $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive $n-$tuple, then we have the following inequalities, which are also well known.

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5. **Cassels' inequality** [10]
If the positive $n-$tuples $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{1.6}$$

where m, M are given, then

$$\frac{\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \leq \frac{(M+m)^2}{4mM}. \tag{1.7}$$

6. **Greub-Reinboldt's inequality** [4]
If $\mathbf{a}$ and $\mathbf{b}$ satisfy the condition (1.1), then

$$\frac{\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{1.8}$$

7. **Generalised Diaz-Metcalf inequality** [1] (see also [6, p. 123])
If $u, v \in [0, 1]$ and $v \leq u$, $u + v = 1$ and (1.6) holds, then one has the inequality

$$u \sum_{k=1}^{n} w_k b_k^2 + vmM \sum_{k=1}^{n} w_k a_k^2 \leq (vm + uM) \sum_{k=1}^{n} w_k a_k b_k. \tag{1.9}$$

8. **Klamkin-McLenaghan's inequality** [5]
If $\mathbf{a}$ and $\mathbf{b}$ satisfy (1.6), then we have the inequality

$$\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2 - \left(\sum_{k=1}^{n} w_k a_k b_k\right)^2 \leq \left(\sqrt{M} - \sqrt{m}\right)^2 \sum_{k=1}^{n} w_k a_k b_k \sum_{k=1}^{n} w_k a_k^2. \tag{1.10}$$

For other reverse results of the (CBS)-inequality, see the recent survey online [3].

The main aim of this paper is to point out some new reverse inequalities of the classical Cauchy-Bunyakovsky-Schwarz result for both real and complex $n-$tuples.

2. Some Reverses of the Cauchy-Bunyakovsky-Schwarz Inequality

The following result holds.

Theorem 1. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$, *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $b_i \neq 0$, $i \in \{1, \dots, n\}$ *and there exists the constant* $\alpha \in \mathbb{K}$ *and* $r > 0$ *such that for any* $k \in \{1, \dots, n\}$

$$\frac{a_k}{b_k} \in \bar{D}(\alpha, r) := \{z \in \mathbb{K} | \, |z - \alpha| \leq r\}, \tag{2.1}$$

then we have the inequality

$$\begin{aligned} \sum_{k=1}^{n} p_k |a_k|^2 + \left(|\alpha|^2 - r^2\right) \sum_{k=1}^{n} p_k |b_k|^2 &\leq 2 \operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] \\ &\leq 2|\alpha| \cdot \left|\sum_{k=1}^{n} p_k a_k b_k\right|. \end{aligned} \tag{2.2}$$

The constant $c = 2$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. From (2.1) we have $\left|a_k - \alpha \bar{b}_k\right|^2 \le r\left|b_k\right|^2$ for each $k \in \{1, \dots, n\}$, which is clearly equivalent to

$$|a_k|^2 + \left(|\alpha|^2 - r^2\right)|b_k|^2 \le 2\operatorname{Re}\left[\bar{\alpha}\left(a_k b_k\right)\right] \tag{2.3}$$

for each $k \in \{1, \dots, n\}$.

Multiplying (2.3) with $p_k \ge 0$ and summing over k from 1 to n, we deduce the first inequality in (1.2). The second inequality is obvious.

To prove the sharpness of the constant 2, assume that under the hypothesis of the theorem there exists a constant $c > 0$ such that

$$\sum_{k=1}^{n} p_k |a_k|^2 + \left(|\alpha|^2 - r^2\right)\sum_{k=1}^{n} p_k |b_k|^2 \le c\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right], \tag{2.4}$$

provided $\frac{a_k}{b_k} \in \bar{D}(\alpha, r)$, $k \in \{1, \dots, n\}$.

Assume that $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $b_1 = b_2 = 1$, $\alpha = r > 0$ and $a_2 = 2r$, $a_1 = 0$. Then $\left|\frac{a_2}{b_2} - \alpha\right| = r$, $\left|\frac{a_1}{b_1} - \alpha\right| = r$ showing that the condition (2.1) holds. For these choices, the inequality (2.4) becomes $2r^2 \le cr^2$, giving $c \ge 2$. ∎

The case where the disk $\bar{D}(\alpha, r)$ does not contain the origin, i.e., $|\alpha| > r$, provides the following interesting reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

Theorem 2. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$ *as in Theorem 1 and assume that* $|\alpha| > r > 0$. *Then we have the inequality*

$$\begin{aligned}\sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 &\le \frac{1}{|\alpha|^2 - r^2}\left\{\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right]\right\}^2 \\ &\le \frac{|\alpha|^2}{|\alpha|^2 - r^2}\left|\sum_{k=1}^{n} p_k a_k b_k\right|^2.\end{aligned} \tag{2.5}$$

The constant $c = 1$ *in the first and second inequality is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Since $|\alpha| > r$, we may divide (2.2) by $\sqrt{|\alpha|^2 - r^2} > 0$ to obtain

$$\begin{aligned}\frac{1}{\sqrt{|\alpha|^2 - r^2}}\sum_{k=1}^{n} p_k |a_k|^2 + \sqrt{|\alpha|^2 - r^2}\sum_{k=1}^{n} p_k |b_k|^2 \\ \le \frac{2}{\sqrt{|\alpha|^2 - r^2}}\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right].\end{aligned} \tag{2.6}$$

On the other hand, by the use of the following elementary inequality

$$\frac{1}{\beta}p + \beta q \ge 2\sqrt{pq} \quad \text{for} \quad \beta > 0 \quad \text{and} \quad p, q \ge 0, \tag{2.7}$$

we may state that

$$2\left(\sum_{k=1}^{n}p_k|a_k|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n}p_k|b_k|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|\alpha|^2-r^2}}\sum_{k=1}^{n}p_k|a_k|^2+\sqrt{|\alpha|^2-r^2}\sum_{k=1}^{n}p_k|b_k|^2. \tag{2.8}$$

Utilising (2.6) and (2.8), we deduce

$$\left(\sum_{k=1}^{n}p_k|a_k|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n}p_k|b_k|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|\alpha|^2-r^2}}\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n}p_ka_kb_k\right)\right],$$

which is clearly equivalent to the first inequality in (2.6).

The second inequality is obvious.

To prove the sharpness of the constant, assume that (2.5) holds with a constant $c>0$, i.e.,

$$\sum_{k=1}^{n}p_k|a_k|^2\sum_{k=1}^{n}p_k|b_k|^2 \leq \frac{c}{|\alpha|^2-r^2}\left\{\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n}p_ka_kb_k\right)\right]\right\}^2 \tag{2.9}$$

provided $\frac{a_k}{b_k}\in\bar{D}(\alpha,r)$ and $|\alpha|>r$.

For $n=2$, $b_2=b_1=1$, $p_1=p_2=\frac{1}{2}$, $a_2,a_1\in\mathbb{R}$, $\alpha,r>0$ and $\alpha>r$, we get from (2.9) that

$$\frac{a_1^2+a_2^2}{2}\leq\frac{c\alpha^2}{\alpha^2-r^2}\left(\frac{a_1+a_2}{2}\right)^2. \tag{2.10}$$

If we choose $a_2=\alpha+r$, $a_1=\alpha-r$, then $|a_i-\alpha|\leq r$, $i=1,2$ and by (2.10) we deduce

$$\alpha^2+r^2\leq\frac{c\alpha^4}{\alpha^2-r^2},$$

which is clearly equivalent to

$$(c-1)\,\alpha^4+r^4\geq 0 \quad\text{for}\quad \alpha>r>0.$$

If in this inequality we choose $\alpha=1$, $r=\varepsilon\in(0,1)$ and let $\varepsilon\to 0+$, then we deduce $c\geq 1$. ∎

The following corollary is a natural consequence of the above theorem.

Corollary 1. *Under the assumptions of Theorem 2, we have the following additive reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$0\leq\sum_{k=1}^{n}p_k|a_k|^2\sum_{k=1}^{n}p_k|b_k|^2-\left|\sum_{k=1}^{n}p_ka_kb_k\right|^2 \leq\frac{r^2}{|\alpha|^2-r^2}\left|\sum_{k=1}^{n}p_ka_kb_k\right|^2. \tag{2.11}$$

The constant $c=1$ is best possible in the sense mentioned above.

Remark 1. *If in Theorem 1, we assume that $|\alpha| = r$, then we obtain the inequality:*

$$\sum_{k=1}^{n} p_k |a_k|^2 \le 2\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] \le 2|\alpha|\left|\sum_{k=1}^{n} p_k a_k b_k\right|. \tag{2.12}$$

The constant 2 is sharp in both inequalities.

We also remark that, if $r > |\alpha|$, then (2.2) may be written as

$$\sum_{k=1}^{n} p_k |a_k|^2 \le \left(r^2 - |\alpha|^2\right)\sum_{k=1}^{n} p_k |b_k|^2 + 2\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] \le \left(r^2 - |\alpha|^2\right)\sum_{k=1}^{n} p_k |b_k|^2 + 2|\alpha|\left|\sum_{k=1}^{n} p_k a_k b_k\right|. \tag{2.13}$$

The following reverse of the Cauchy-Bunyakovsky-Schwarz inequality also holds.

Theorem 3. *Let* **a**, **b**, **p** *be as in Theorem 1 and assume that $\alpha \in \mathbb{K}$, $\alpha \neq 0$ and $r > 0$. Then we have the inequalities*

$$\begin{aligned} 0 &\le \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} p_k |b_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k a_k b_k\right| \\ &\le \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} p_k |b_k|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\alpha}}{|\alpha|}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] \\ &\le \frac{1}{2} \cdot \frac{r^2}{|\alpha|}\sum_{k=1}^{n} p_k |b_k|^2. \end{aligned} \tag{2.14}$$

The constant $\frac{1}{2}$ is best possible in the sense mentioned above.

Proof. From Theorem 1, we have

$$\sum_{k=1}^{n} p_k |a_k|^2 + |\alpha|^2 \sum_{k=1}^{n} p_k |b_k|^2 \le 2\operatorname{Re}\left[\bar{\alpha}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] + r^2 \sum_{k=1}^{n} p_k |b_k|^2. \tag{2.15}$$

Since $\alpha \neq 0$, we can divide (2.15) by $|\alpha|$, getting

$$\frac{1}{|\alpha|}\sum_{k=1}^{n} p_k |a_k|^2 + |\alpha| \sum_{k=1}^{n} p_k |b_k|^2 \le 2\operatorname{Re}\left[\frac{\bar{\alpha}}{|\alpha|}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] + \frac{r^2}{|\alpha|}\sum_{k=1}^{n} p_k |b_k|^2. \tag{2.16}$$

Utilising the inequality (2.7), we may state that

$$2\left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} p_k |b_k|^2\right)^{\frac{1}{2}} \le \frac{1}{|\alpha|}\sum_{k=1}^{n} p_k |a_k|^2 + |\alpha| \sum_{k=1}^{n} p_k |b_k|^2. \tag{2.17}$$

Making use of (2.16) and (2.17), we deduce the second inequality in (2.14).

The first inequality in (2.14) is obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that there exists a $c > 0$ such that

$$\left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} p_k |b_k|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\alpha}}{|\alpha|}\left(\sum_{k=1}^{n} p_k a_k b_k\right)\right] \leq c \cdot \frac{r^2}{|\alpha|} \sum_{k=1}^{n} p_k |b_k|^2, \tag{2.18}$$

provided $\left|\frac{a_k}{b_k} - \alpha\right| \leq r$, $\alpha \neq 0$, $r > 0$.

If we choose $n = 2$, $\alpha > 0$, $b_1 = b_2 = 1$, $a_1 = \alpha + r$, $a_2 = \alpha - r$, then from (2.18) we deduce

$$\sqrt{r^2 + \alpha^2} - \alpha \leq c \frac{r^2}{\alpha}. \tag{2.19}$$

If we multiply (2.19) with $\sqrt{r^2 + \alpha^2} + \alpha > 0$ and then divide it by $r > 0$, we deduce

$$1 \leq \frac{\sqrt{r^2 + \alpha^2} + \alpha}{\alpha} \cdot c \tag{2.20}$$

for any $r > 0$, $\alpha > 0$.

If in (2.20) we let $r \to 0+$, then we get $c \geq \frac{1}{2}$, and the sharpness of the constant is proved. ∎

3. A Cassels Type Inequality for Complex Numbers

The following result holds.

Theorem 4. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$, *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $b_i \neq 0$, $i \in \{1, \dots, n\}$ *and there exist the constants* $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ *and* $\Gamma \neq \gamma$, *so that either*

$$\left|\frac{a_k}{b_k} - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{3.1}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\Gamma - \frac{a_k}{b_k}\right)\left(\frac{\overline{a_k}}{b_k} - \bar{\gamma}\right)\right] \geq 0 \quad \text{for each} \quad k \in \{1, \dots, n\} \tag{3.2}$$

holds, then we have the inequalities

$$\sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leq \frac{1}{2\operatorname{Re}(\Gamma\bar{\gamma})}\left\{\operatorname{Re}\left[(\bar{\gamma} + \bar{\Gamma})\sum_{k=1}^{n} p_k a_k b_k\right]\right\}^2 \leq \frac{|\Gamma + \gamma|^2}{4\operatorname{Re}(\Gamma\bar{\gamma})}\left|\sum_{k=1}^{n} p_k a_k b_k\right|^2. \tag{3.3}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *are best possible in (3.3).*

Proof. The fact that the relations (3.1) and (3.2) are equivalent follows by the simple fact that for $z, u, U \in \mathbb{C}$, the following inequalities are equivalent

$$\left|z - \frac{u + U}{2}\right| \leq \frac{1}{2}|U - u|$$

and

$$\operatorname{Re}\left[(u-z)\left(\bar{z}-\bar{u}\right)\right] \geq 0.$$

Define $\alpha = \frac{\gamma+\Gamma}{2}$ and $r = \frac{1}{2}\left|\Gamma-\gamma\right|$. Then

$$\left|\alpha\right|^2 - r^2 = \frac{\left|\Gamma+\gamma\right|^2}{4} - \frac{\left|\Gamma-\gamma\right|^2}{4} = \operatorname{Re}\left(\Gamma\bar{\gamma}\right) > 0.$$

Consequently, we may apply Theorem 2, and the inequalities (3.3) are proved.

The sharpness of the constants may be proven in a similar way to that in the proof of Theorem 2, and we omit the details. ∎

The following additive version also holds.

Corollary 2. *With the assumptions in Theorem 4, we have*

$$\sum_{k=1}^{n} p_k \left|a_k\right|^2 \sum_{k=1}^{n} p_k \left|b_k\right|^2 - \left|\sum_{k=1}^{n} p_k a_k b_k\right|^2 \leq \frac{\left|\Gamma-\gamma\right|^2}{4\operatorname{Re}\left(\Gamma\bar{\gamma}\right)} \left|\sum_{k=1}^{n} p_k a_k b_k\right|^2. \tag{3.4}$$

The constant $\frac{1}{4}$ is also best possible.

Remark 2. *With the above assumptions and if* $\operatorname{Re}\left(\Gamma\bar{\gamma}\right) = 0$, *then by the use of Remark 1, we may deduce the inequality*

$$\sum_{k=1}^{n} p_k \left|a_k\right|^2 \leq \operatorname{Re}\left[\left(\bar{\gamma}+\bar{\Gamma}\right) \sum_{k=1}^{n} p_k a_k b_k\right] \leq \left|\Gamma+\gamma\right| \left|\sum_{k=1}^{n} p_k a_k b_k\right|. \tag{3.5}$$

If $\operatorname{Re}\left(\Gamma\bar{\gamma}\right) < 0$, *then, by Remark 1, we also have*

$$\begin{aligned} \sum_{k=1}^{n} p_k \left|a_k\right|^2 &\leq -\operatorname{Re}\left(\Gamma\bar{\gamma}\right) \sum_{k=1}^{n} p_k \left|b_k\right|^2 + \operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right) \sum_{k=1}^{n} p_k a_k b_k\right] \\ &\leq -\operatorname{Re}\left(\Gamma\bar{\gamma}\right) \sum_{k=1}^{n} p_k \left|b_k\right|^2 + \left|\Gamma+\gamma\right| \left|\sum_{k=1}^{n} p_k a_k b_k\right|. \end{aligned} \tag{3.6}$$

Remark 3. *If $a_k, b_k > 0$ and there exist the constants $m, M > 0$ $(M > m)$ with*

$$m \leq \frac{a_k}{b_k} \leq M \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{3.7}$$

then, obviously (3.1) holds with $\gamma = m$, $\Gamma = M$, also $\Gamma\bar{\gamma} = Mm > 0$ and by (3.3) we deduce

$$\sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 \leq \frac{(M+m)^2}{4mM} \left(\sum_{k=1}^{n} p_k a_k b_k\right)^2, \tag{3.8}$$

that is, Cassels' inequality.

4. A Shisha-Mond Type Inequality for Complex Numbers

The following result holds.

Theorem 5. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$, *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $b_i \neq 0$, $i \in \{1, \dots, n\}$ *and there exist the constants* $\gamma, \Gamma \in \mathbb{K}$ *such that* $\Gamma \neq \gamma, -\gamma$ *and either*

$$\left|\frac{a_k}{\overline{b_k}} - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right| \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{4.1}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\Gamma-\frac{a_k}{b_k}\right)\left(\frac{\overline{a_k}}{b_k}-\bar{\gamma}\right)\right]\geq 0 \quad \textit{for each} \quad k\in\{1,\dots,n\}, \tag{4.2}$$

holds, then we have the inequalities

$$\begin{aligned} 0 &\leq \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n} p_k |b_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k a_k b_k\right| \\ &\leq \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n} p_k |b_k|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\sum_{k=1}^{n} p_k a_k b_k\right] \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\sum_{k=1}^{n} p_k |b_k|^2 . \end{aligned} \tag{4.3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Follows by Theorem 3 on choosing $\alpha=\frac{\gamma+\Gamma}{2}\neq 0$ and $r=\frac{1}{2}|\Gamma-\gamma|>0$.

The proof for the best constant follows in a similar way to that in the proof of Theorem 3 and we omit the details. ∎

Remark 4. *If* $a_k, b_k > 0$ *and there exists the constants* $m, M > 0$ $(M > m)$ *with*

$$m\leq\frac{a_k}{b_k}\leq M \quad \textit{for each} \quad k\in\{1,\dots,n\}, \tag{4.4}$$

then we have the inequality

$$\begin{aligned} 0 &\leq \left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^{n} p_k a_k b_k \\ &\leq \frac{1}{4}\cdot\frac{(M-m)^2}{(M+m)}\sum_{k=1}^{n} p_k b_k^2. \end{aligned} \tag{4.5}$$

The constant $\frac{1}{4}$ *is best possible. For* $p_k=\frac{1}{n}$, $k\in\{1,\dots,n\}$, *we recapture the result from* [2, Theorem 5.21] *that has been obtained from a reverse inequality due to Shisha and Mond* [8].

5. Further Reverses of the (CBS)-Inequality

The following result holds.

Theorem 6. *Let* $\mathbf{a}=(a_1,\dots,a_n)$, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{K}^n$ *and* $r>0$ *such that for* $p_i\geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$

$$\sum_{i=1}^{n} p_i |b_i-\overline{a_i}|^2 \leq r^2 < \sum_{i=1}^{n} p_i |a_i|^2 . \tag{5.1}$$

Then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left| \sum_{i=1}^{n} p_i a_i b_i \right|^2 \tag{5.2}$$
$$\leq \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[\operatorname{Re} \left(\sum_{i=1}^{n} p_i a_i b_i \right) \right]^2$$
$$\leq r^2 \sum_{i=1}^{n} p_i |b_i|^2 .$$

The constant $c = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. From the first condition in (5.1), we have

$$\sum_{i=1}^{n} p_i \left[|b_i|^2 - 2 \operatorname{Re} (b_i a_i) + |a_i|^2 \right] \leq r^2,$$

giving

$$\sum_{i=1}^{n} p_i |b_i|^2 + \sum_{i=1}^{n} p_i |a_i|^2 - r^2 \leq 2 \operatorname{Re} \left(\sum_{i=1}^{n} p_i a_i b_i \right). \tag{5.3}$$

Since, by the second condition in (5.1) we have

$$\sum_{i=1}^{n} p_i |a_i|^2 - r^2 > 0,$$

we may divide (5.3) by $\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2} > 0$, getting

$$\frac{\sum_{i=1}^{n} p_i |b_i|^2}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2} \leq \frac{2 \operatorname{Re} \left(\sum_{i=1}^{n} p_i a_i b_i \right)}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}}. \tag{5.4}$$

Utilising the elementary inequality

$$\frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq} \quad \text{for} \quad p, q \geq 0 \quad \text{and} \quad \alpha > 0, \tag{5.5}$$

we may write that

$$2\sqrt{\sum_{i=1}^{n} p_i |b_i|^2} \leq \frac{\sum_{i=1}^{n} p_i |b_i|^2}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}. \tag{5.6}$$

Combining (5.5) with (5.6) we deduce

$$\sqrt{\sum_{i=1}^{n} p_i |b_i|^2} \leq \frac{\operatorname{Re} \left(\sum_{i=1}^{n} p_i a_i b_i \right)}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}}. \tag{5.7}$$

Taking the square in (5.7), we obtain

$$\sum_{i=1}^{n} p_i |b_i|^2 \left(\sum_{i=1}^{n} p_i |a_i|^2 - r^2 \right) \leq \left[\operatorname{Re} \left(\sum_{i=1}^{n} p_i a_i b_i \right) \right]^2,$$

giving the third inequality in (5.2).

The other inequalities are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that there exists a constant $c>0$ such that

$$\sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right)\right]^2 \leq cr^2 \sum_{i=1}^{n} p_i |b_i|^2, \tag{5.8}$$

provided

$$\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 \leq r^2 < \sum_{i=1}^{n} p_i |a_i|^2 .$$

Let $r=\sqrt{\varepsilon}$, $\varepsilon \in (0,1)$, $a_i, e_i \in \mathbb{C}$, $i \in \{1,\dots,n\}$ with $\sum_{i=1}^{n} p_i |a_i|^2 = \sum_{i=1}^{n} p_i |e_i|^2 = 1$ and $\sum_{i=1}^{n} p_i a_i e_i = 0$. Put $b_i = \overline{a_i} + \sqrt{\varepsilon} e_i$. Then, obviously

$$\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 = r^2, \qquad \sum_{i=1}^{n} p_i |a_i|^2 = 1 > r$$

and

$$\sum_{i=1}^{n} p_i |b_i|^2 = \sum_{i=1}^{n} p_i |a_i|^2 + \varepsilon \sum_{i=1}^{n} p_i |e_i|^2 = 1+\varepsilon,$$

$$\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right) = \sum_{i=1}^{n} p_i |a_i|^2 = 1$$

and thus

$$\sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right)\right]^2 = \varepsilon.$$

Using (5.8), we may write

$$\varepsilon \leq c\varepsilon (1+\varepsilon) \quad \text{for} \quad \varepsilon \in (0,1),$$

giving $1 \leq c(1+\varepsilon)$ for $\varepsilon \in (0,1)$. Making $\varepsilon \to 0+$, we deduce $c \geq 1$. ∎

The following result also holds.

Theorem 7. *Let* $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1,\dots,p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $\gamma, \Gamma \in \mathbb{K}$ *such that* $\operatorname{Re}(\gamma\bar{\Gamma}) > 0$ *and either*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(\Gamma \overline{y_i} - x_i)(\overline{x_i} - \bar{\gamma} y_i)\right] \geq 0, \tag{5.9}$$

or, equivalently,

$$\sum_{i=1}^{n} p_i \left| x_i - \frac{\gamma+\Gamma}{2} \cdot \overline{y_i} \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 . \tag{5.10}$$

Then we have the inequalities

$$\begin{aligned}\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 &\leq \frac{1}{4} \cdot \frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right) \sum_{i=1}^{n} p_i x_i y_i\right]\right\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{i=1}^{n} p_i x_i y_i \right|^2 .\end{aligned} \tag{5.11}$$

The constant $\frac{1}{4}$ *is best possible in both inequalities.*

Proof. Define $b_i = x_i$ and $a_i = \frac{\bar{\Gamma}+\bar{\gamma}}{2} \cdot y_i$ and $r = \frac{1}{2}|\Gamma - \gamma| \left(\sum_{i=1}^n p_i |y_i|^2\right)^{\frac{1}{2}}$. Then, by (5.10)

$$\begin{aligned}\sum_{i=1}^n p_i |b_i - \overline{a_i}|^2 &= \sum_{i=1}^n p_i \left| x_i - \frac{\gamma+\Gamma}{2} \cdot \overline{y_i} \right|^2 \\ &\le \frac{1}{4}|\Gamma-\gamma|^2 \sum_{i=1}^n p_i |y_i|^2 = r^2,\end{aligned}$$

showing that the first condition in (5.1) is fulfilled.

We also have

$$\begin{aligned}\sum_{i=1}^n p_i |a_i|^2 - r^2 &= \sum_{i=1}^n p_i \left|\frac{\Gamma+\gamma}{2}\right|^2 |y_i|^2 - \frac{1}{4}|\Gamma-\gamma|^2 \sum_{i=1}^n p_i |y_i|^2 \\ &= \mathrm{Re}(\Gamma\bar{\gamma}) \sum_{i=1}^n p_i |y_i|^2 > 0\end{aligned}$$

since $\mathrm{Re}\left(\gamma\bar{\Gamma}\right) > 0$, and thus the condition in (5.1) is also satisfied.

Using the second inequality in (5.2), one may write

$$\begin{aligned}&\sum_{i=1}^n p_i \left|\frac{\Gamma+\gamma}{2}\right|^2 |y_i|^2 \sum_{i=1}^n p_i |x_i|^2 - \left[\mathrm{Re} \sum_{i=1}^n p_i \left(\frac{\bar{\Gamma}+\bar{\gamma}}{2}\right) y_i x_i\right]^2 \\ \le\ &\frac{1}{4}|\Gamma-\gamma|^2 \sum_{i=1}^n p_i |y_i|^2 \sum_{i=1}^n p_i |x_i|^2,\end{aligned}$$

giving

$$\frac{|\Gamma+\gamma|^2 - |\Gamma-\gamma|^2}{4} \sum_{i=1}^n p_i |y_i|^2 \sum_{i=1}^n p_i |x_i|^2 \le \frac{1}{4} \mathrm{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right) \sum_{i=1}^n p_i x_i y_i\right]^2,$$

which is clearly equivalent to the first inequality in (5.11).

The second inequality in (5.11) is obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that the first inequality in (5.11) holds with a constant $C > 0$, i.e.,

$$\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2 \le C \cdot \frac{\left\{\mathrm{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\sum_{i=1}^n p_i x_i y_i\right]\right\}^2}{\mathrm{Re}(\Gamma\bar{\gamma})}, \tag{5.12}$$

provided $\mathrm{Re}\left(\gamma\bar{\Gamma}\right) > 0$ and either (5.9) or (5.10) holds.

Assume that $\Gamma, \gamma > 0$ and let $x_i = \gamma \bar{y}_i$. Then (5.9) holds true and by (5.12) we deduce

$$\gamma^2 \left(\sum_{i=1}^n p_i |y_i|^2\right)^2 \le C \frac{(\Gamma+\gamma)^2 \gamma^2 \left(\sum_{i=1}^n p_i |y_i|^2\right)^2}{\Gamma\gamma},$$

giving

$$\Gamma\gamma \le C(\Gamma+\gamma)^2 \quad \text{for any} \quad \Gamma, \gamma > 0. \tag{5.13}$$

Let $\varepsilon \in (0,1)$ and choose in (5.13) $\Gamma = 1+\varepsilon$, $\gamma = 1-\varepsilon > 0$ to get $1-\varepsilon^2 \leq 4C$ for any $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0+$, we deduce $C \geq \frac{1}{4}$ and the sharpness of the constant is proved.

Finally, we note that the conditions (5.9) and (5.10) are equivalent since in an inner product space $(H, \langle \cdot, \cdot \rangle)$ for any vectors $x, z, Z \in H$ one has $\operatorname{Re}\langle Z-x, x-z\rangle \geq 0$ iff $\left\| x - \frac{z+Z}{2} \right\| \leq \frac{1}{2}\|Z-z\|$ [1]. We omit the details. ∎

6. More Reverses of the (CBS)-Inequality

The following result holds.

Theorem 8. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^n p_i = 1$. *If* $r > 0$ *and the following condition is satisfied*

$$\sum_{i=1}^n p_i \left| b_i - \overline{a_i} \right|^2 \leq r^2, \tag{6.1}$$

then we have the inequalities

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n p_i a_i b_i \right| \\ &\leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^n p_i \operatorname{Re}(a_i b_i) \right| \\ &\leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}(a_i b_i) \\ &\leq \frac{1}{2} r^2. \end{aligned} \tag{6.2}$$

The constant $\frac{1}{2}$ *is best possible in (6.2) in the sense that it cannot be replaced by a smaller constant.*

Proof. The condition (6.1) is clearly equivalent to

$$\sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2 \leq 2 \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) + r^2. \tag{6.3}$$

Using the elementary inequality

$$2 \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2 \tag{6.4}$$

and (6.3), we deduce

$$2 \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} \leq 2 \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) + r^2, \tag{6.5}$$

giving the last inequality in (6.2). The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that

$$0 \leq \left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) \leq c r^2 \tag{6.6}$$

for any $\mathbf{a}$, $\mathbf{b} \in \mathbb{K}^n$ and $r > 0$ satisfying (6.1).

Assume that $\mathbf{a}$, $\bar{\mathbf{e}} \in H$, $\bar{\mathbf{e}} = (e_1, \dots, e_n)$ with $\sum_{i=1}^n p_i |a_i|^2 = \sum_{i=1}^n p_i |e_i|^2 = 1$ and $\sum_{i=1}^n p_i a_i e_i = 0$. If $r = \sqrt{\varepsilon}$, $\varepsilon > 0$, and if we define $\mathbf{b} = \bar{\mathbf{a}} + \sqrt{\varepsilon}\bar{\mathbf{e}}$ where $\bar{\mathbf{a}} = (\overline{a_1}, \dots, \overline{a_n}) \in \mathbb{K}^n$, then $\sum_{i=1}^n p_i |b_i - \overline{a_i}|^2 = \varepsilon = r^2$, showing that the condition (6.1) is fulfilled.

On the other hand,

$$\begin{aligned}
&\left(\sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2\right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}(b_i a_i) \\
&= \left(\sum_{i=1}^n p_i \left|\overline{a_i} + \sqrt{\varepsilon} e_i\right|^2\right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}\left[\left(\overline{a_i} + \sqrt{\varepsilon} e_i\right) a_i\right] \\
&= \left(\sum_{i=1}^n p_i |a_i|^2 + \varepsilon \sum_{i=1}^n |e_i|^2\right)^{\frac{1}{2}} - \sum_{i=1}^n p_i |a_i|^2 \\
&= \sqrt{1+\varepsilon} - 1.
\end{aligned}$$

Utilizing (6.6), we conclude that

$$\sqrt{1+\varepsilon} - 1 \le c\varepsilon \quad \text{for any} \quad \varepsilon > 0. \tag{6.7}$$

Multiplying (6.7) by $\sqrt{1+\varepsilon} + 1 > 0$ and thus dividing by $\varepsilon > 0$, we get

$$\left(\sqrt{1+\varepsilon} - 1\right) c \ge 1 \quad \text{for any} \quad \varepsilon > 0. \tag{6.8}$$

Letting $\varepsilon \to 0+$ in (6.8), we deduce $c \ge \frac{1}{2}$, and the theorem is proved. ∎

Finally, the following result also holds.

Theorem 9. *Let* $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^n p_i = 1$, *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\Gamma \ne \gamma, -\gamma$, *so that either*

$$\sum_{i=1}^n p_i \operatorname{Re}\left[\left(\Gamma \overline{y_i} - x_i\right)\left(\overline{x_i} - \bar{\gamma} y_i\right)\right] \ge 0, \tag{6.9}$$

or, equivalently,

$$\sum_{i=1}^n p_i \left|x_i - \frac{\gamma + \Gamma}{2} \cdot \overline{y_i}\right|^2 \le \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^n p_i |y_i|^2 \tag{6.10}$$

holds. Then we have the inequalities

$$\begin{aligned}
0 &\le \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n p_i x_i y_i\right| \\
&\le \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n p_i \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i\right]\right| \\
&\le \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2\right)^{\frac{1}{2}} - \sum_{i=1}^n p_i \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i\right] \\
&\le \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^n p_i |y_i|^2 .
\end{aligned} \tag{6.11}$$

The constant $\frac{1}{4}$ in the last inequality is best possible.

Proof. Consider $b_i = x_i$, $a_i = \frac{\bar{\Gamma}+\bar{\gamma}}{2} \cdot y_i$, $i \in \{1, \dots, n\}$ and

$$r := \frac{1}{2}(\Gamma - \gamma)\left(\sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}}.$$

Then, by (6.10), we have

$$\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 = \sum_{i=1}^{n} p_i \left|x_i - \frac{\gamma + \Gamma}{2} \cdot y_i\right|^2 \le \frac{1}{4}|\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 = r^2$$

showing that (6.1) is valid.

By the use of the last inequality in (6.2), we have

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i \left|\frac{\Gamma + \gamma}{2}\right|^2 |y_i|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} p_i \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{2} x_i y_i\right] \\ &\le \frac{1}{8}|\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2. \end{aligned}$$

Dividing by $\frac{1}{2}|\Gamma + \gamma| > 0$, we deduce

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} p_i \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i\right] \\ &\le \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{n} p_i |y_i|^2, \end{aligned}$$

which is the last inequality in (6.11).

The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that there exists a constant $c > 0$, such that

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} p_i \operatorname{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} x_i y_i\right] \\ \le\ & c \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{n} p_i |y_i|^2, \end{aligned} \tag{6.12}$$

provided either (6.9) or (6.10) holds.

Let $n = 2$, $\mathbf{y} = (1, 1)$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\Gamma, \gamma > 0$ with $\Gamma > \gamma$. Then by (6.12) we deduce

$$\sqrt{2}\sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \le 2c\frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}. \tag{6.13}$$

If $x_1 = \Gamma$, $x_2 = \gamma$, then $(\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0$, showing that the condition (6.9) is valid for $n = 2$ and $\mathbf{p}$, $\mathbf{x}$, $\mathbf{y}$ as above. Replacing x_1 and x_2 in (6.13), we deduce

$$\sqrt{2}\sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \le 2c\frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}. \tag{6.14}$$

If in (6.14) we choose $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon$ with $\varepsilon \in (0, 1)$, we deduce

$$\sqrt{1 + \varepsilon^2} - 1 \le 2c\varepsilon^2. \tag{6.15}$$

Finally, multiplying (6.15) with $\sqrt{1 + \varepsilon^2} + 1 > 0$ and then dividing by ε^2, we deduce

$$1 \le 2c\left(\sqrt{1 + \varepsilon^2} + 1\right) \quad \text{for any} \quad \varepsilon > 0. \tag{6.16}$$

Letting $\varepsilon \to 0+$ in (6.16), we get $c \ge \frac{1}{4}$, and the sharpness of the constant is proved. ∎

Remark 5. *The integral version may be stated in a canonical way. The corresponding inequalities for integrals will be considered in another work devoted to positive linear functionals with complex values that is in preparation.*

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