ON WEITZENBOECK'S INEQUALITY AND ITS GENERALIZATIONS

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Abstract. In this paper, a new proof of the equivalence for Weitzenboeck's inequality and Finsler-Hadwiger's inequality is given, and some generalizations of Weitzenboeck's inequality for only one triangle is proved.

1. Introduction

Throughout the paper we assume a, b, c denote the opposite sides of triangle ABC, A, B, C the angles, s the semi-perimeter, Δ the area and R the circumradius. Moreover, we will customarily use the cyclic sum symbol and cyclic product symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, $\sum f(a,b) = f(a,b) + f(b,c) + f(c,a)$ and $\prod f(a) = f(a)f(b)f(c)$, similarly, one defines others.

In 1919, Weitzenboeck [1] obtained the following interesting inequality for sides and area of the triangle

$$\sum a^2 \geqslant 4\sqrt{3}\Delta \tag{1.1}$$

with equality holding if and only if the triangle ABC is the equilateral one.

Inequality (1.1) is called Weitzenboeck's inequality. In the theory of geometric inequality, Weitzenboeck's inequality and its generalizations often play fundamental role, these results are interesting and useful. For example:

In 1937, P.Finsler and H.Hadwiger [2] first studied the generalization of Weitzenboeck's inequality, their got:

$$\sum a^2 \geqslant 4\sqrt{3}\Delta + \sum (a-b)^2 \tag{1.2}$$

with equality holding if only and if the triangle ABC is equilateral.

A weighted representation for Weitzenboeck's inequality is given in [3], A.George verified that

$$\sum \left(\frac{\alpha_1}{\alpha_2 + \alpha_3}\, a^2 \right) \geqslant 2\sqrt{3}\Delta \tag{1.3}$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$.

A.Oppenheim [4] published the following useful weighted formula:

$$(\sum \lambda_1 a^2)^2 \geqslant 16\Delta^2 (\sum \lambda_1 \lambda_2) \tag{1.4}$$

with equality holding if and only if $\lambda_1 : \lambda_2 : \lambda_3 = (b^2 + c^2 - a^2) : (c^2 + a^2 - b^2) : (a^2 + b^2 - c^2)$, where $\lambda_1, \lambda_2, \lambda_3$ are the real numbers.

For two triangles, the generalization of Weitzenboeck's inequality (1.1) is the well-known Neuberg-Pedoe inequality:

$$\sum a_1^2 (b_2^2 + c_2^2 - a_2^2) \geqslant 16\Delta_1\Delta_2 \tag{1.5}$$

with equality holding if and only if $\Delta A_1B_1C_1 \backsim \Delta A_2B_2C_2$.

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In this paper, a new proof of the equivalence for Weitzenboeck's inequality (1.1) and Finsler-Hadwiger's inequality (1.2) is given, and some generalizations of Weitzenboeck's inequality for only one triangle is proved.

2. A New Proof of the Equivalence for Weitzenboeck's Inequality (1.1) and Finsler-Hadwiger's Inequality (1.2)

To prove the equivalence for Weitzenboeck's inequality (1.1) and Finsler-Hadwiger's inequality (1.2), the following Lemma2.1 will be used.

Lemma 2.1. *In every triangle ABC, if*

$$a_1 = \sqrt{a(b+c-a)}, b_1 = \sqrt{b(c+a-b)}, c_1 = \sqrt{c(a+b-c)} \tag{2.1}$$

and

$$A_1 = (\pi - A)/2, B_1 = (\pi - B)/2, C_1 = (\pi - C)/2, \tag{2.2}$$

then the triangle $A'B'C'$ for sides a_1, b_1, c_1 is a triangle for angles A_1, B_1, C_1, and the area $\triangle_1 = \triangle$.

Proof. Because $a+b-c>0, b+c>a, c+a-b>0$, and from the expansion (2.1), we have

$$\begin{aligned}(a_1+b_1)^2 &= a(b+c-a)+b(c+a-b)+2\sqrt{a(b+c-a)b(c+a-b)}\\ &> a(b+c-a)+b(c+a-b) = c(a+b)-(a-b)^2\\ &> c(a+b)-c^2 = c(a+b-c) = c_1^2\end{aligned}$$

that is

$$a_1 + b_1 > c_1$$

similarly, we can obtain $b_1+c_1>a_1, c_1+a_1>b_1$. Therefore, a_1, b_1, c_1 are three sides of a triangle.

According to the expansion (2.2) and $A_1+B_1+C_1 = (\pi-A)/2+(\pi-B)/2+(\pi-C)/2 = \pi$, we get that A_1, B_1, C_1 are three angles of a triangle.

Also, assume a triangle for the sides a_1, b_1, c_1 is $\Delta A'B'C'$, then utilizing the fact that

$$\cos A' = \frac{b_1^2+c_1^2-c_1^2}{2b_1c_1} = \frac{b(c+a-b)+c(a+b-c)-a(b+c-a)}{2\sqrt{b(c+a-b)}\sqrt{c(a+b-c)}} = \sin\frac{A}{2} = \cos\frac{\pi-A}{2} = \cos A_1,$$

we immediately get $A' = A_1$. Similarly, we can obtain $B' = B_1, C' = C_1$. That is the triangle $A'B'C'$ for sides a_1, b_1, c_1 is a triangle for angles A_1, B_1, C_1.

Finally, we have

$$16\Delta_1^2 = 2\sum a_1^2b_1^2 - \sum a_1^4 = 2\sum a^2b^2 - \sum a^4 = 16\Delta^2,$$

i.e., $\Delta_1 = \Delta$. The proof of Lemma2.1 is completed. ∎

Theorem 2.1. *Weitzenboeck's inequality* (1.1) *and Finsler-Hadwiger's inequality* (1.2) *are equivalence.*

Proof. Firstly, Finsler-Hadwiger's inequality (1.2) $\Longrightarrow$ Weitzenboeck's inequality (1.1) is obvious.

Secondly, to prove Weitzenboeck's inequality (1.1) $\Longrightarrow$ Finsler-Hadwiger's inequality (1.2).

From Weitzenboeck's inequality (1.1) and Lemma2.1, we obtain

$$\sum a_1^2 \geqslant 4\sqrt{3}\Delta_1,$$

and

$$\sum a(b+c-a) \geqslant 4\sqrt{3}\Delta$$

that is Finsler-Hadwiger's inequality (1.2). Theorem2.1 is proved. ∎

Remark 2.1. *By the same way, we can prove that the Neuberg-Pedoe inequality* (1.5) *and the following Zh.-P An's inequality* (2.3)*(see also* [14]*) are equivalence:*

$$\sum a_1(b_1+c_1-a_1)(c_2+a_2-b_2)(a_2+b_2-c_2) \geqslant 16\Delta_1\Delta_2 \tag{2.3}$$

with equality holding if and only if $\Delta A_1B_1C_1 \backsim \Delta A_2B_2C_2$.

By using Lemma2.1 and Finsler-Hadwiger's inequality (1.2),we easily prove the following corollary:

Corollary 2.1. *In every triangleABC, we have*

$$\sum a^2 \geqslant 4\sqrt{3}\Delta + \sum (a-b)^2 + \sum \left(\sqrt{a(b+c-a)} - \sqrt{b(c+a-b)}\right)^2 \tag{2.4}$$

with equality holding if only and if the triangle ABC is the equilateral triangle.

Theorem 2.2. *Assume μ is a real number, then in every triangle ABC, the inequality*

$$\sum a^2 \geqslant 4\sqrt{3}\Delta + \mu \sum (a-b)^2 \tag{2.5}$$

holds that the best possible is the coefficient $\mu = 1$.

Proof. In (2.5), let $a = b = 1, c = t$, then we have

$$2 + t^2 \geqslant \sqrt{3}t\sqrt{4-t^2} + 2\mu(1-t^2)$$

Set $t \to 0$, we obtain $\mu \leqslant 1$, therefore the coefficient $\mu = 1$ is the best possible. ∎

3. Some Generalized Results for Triangle

In this section, we will list another generalizations of Weitzenboeck's inequality for triangle.

Theorem 3.1. *If one of $\lambda_1+\lambda_2, \lambda_2+\lambda_3$ and $\lambda_3+\lambda_1$ greater then zero, and $\sum \lambda_1\lambda_2 > 0$, in every triangle ABC we have*

$$\sum \lambda_1 a^2 \geqslant 4\sqrt{\sum \lambda_1\lambda_2}\ \Delta + \left(\sqrt{\lambda_3+\lambda_1}\ a - \sqrt{\lambda_2+\lambda_3}\ b\right)^2 \tag{3.1}$$

with equality holding if and only if $\angle C = \arccos(\lambda_3/\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)})$.

Proof. From one of $\lambda_1+\lambda_2, \lambda_2+\lambda_3$ and $\lambda_3+\lambda_1$ greater then zero, and $\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 > 0$, we have another two of $\lambda_1+\lambda_2, \lambda_2+\lambda_3$ and $\lambda_3+\lambda_1$ greater then zero, and

$$\left| \frac{\lambda_3}{\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}} \right| < 1.$$

Let

$$\theta = \arccos \frac{\lambda_3}{\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}},$$

then

$$\sin\theta = \frac{\sqrt{\sum \lambda_1\lambda_2}}{\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}},\ \cos\theta = \frac{\lambda_3}{\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}}.$$

From $1 \geqslant \cos(C-\theta)$, and $2ab\cos C = a^2+b^2-c^2,\ 2ab\sin C = 4\Delta$, we obtain

$$1 \geqslant \frac{\lambda_3}{\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}}\cos C + \frac{\sqrt{\sum \lambda_1\lambda_2}}{\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}}\sin C$$

or

$$2\sqrt{(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)}ab \geqslant \lambda_3(a^2+b^2-c^2) + 4\sqrt{\sum \lambda_1\lambda_2}\Delta$$

that is inequality (3.1), with the equality holding if and only if $1 = \cos(C - \theta)$, i.e.

$$\angle C = \arccos \frac{\lambda_3}{\sqrt{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}}.$$

The proof is completed. ∎

Theorem3.1 is obtained by K.-Ch. Yang in [5]. We can give some results for Theorem3.1, its not only among inequality (1.4), but also get some another corollaries:

Corollary 3.1. *In every triangle, we have*

$$\sum a^2 \geqslant 4\sqrt{3}\Delta + 2(a-b)^2 \tag{3.2}$$

with equality holding if and only if $\angle C = \pi/3$.

Corollary 3.2. *In every triangle ABC, we have*

$$\sum a^2 \geqslant 4\sqrt{3}\Delta + \sum (a-b)^2 + 2\left(\sqrt{a(b+c-a)} - \sqrt{b(c+a-b)}\ \right)^2 \tag{3.3}$$

with equality holding if only and if $\angle C = \pi/3$.

Corollary 3.3. (Pólya-Szegö[6]) *In every triangle ABC, we have*

$$\frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}} \geqslant \Delta \tag{3.4}$$

with equality holding if and only if the triangle ABC is equilateral.

From Lemma2.1, we easily obtain

Lemma 3.1. *In every triangle ABC, if the sequence of* $\{\Delta A_k B_k C_k\}_{k=0}^n$ *that the sides and angles respectively as follow*

$$\begin{aligned} a_k &= \sqrt{a_{k-1}(b_{k-1} + c_{k-1} - a_{k-1})}, \\ b_k &= \sqrt{b_{k-1}(c_{k-1} + a_{k-1} - b_{k-1})}, \\ c_k &= \sqrt{c_{k-1}(a_{k-1} + b_{k-1} - c_{k-1})} \end{aligned} \tag{3.5}$$

and

$$A_k = (\pi - A_{k-1})/2, B_k = (\pi - B_{k-1})/2, C_k = (\pi - C_{k-1})/2, \tag{3.6}$$

where $\Delta A_0 B_0 C_0$ *is the triangle ABC. Then the triangle for sides* a_k, b_k, c_k *is a triangle for angles* A_k, B_k, C_k, *and we have*

$$A_k = [(2^k - (-1)^k)(\pi/3) + (-1)^k A]/2^k \tag{3.7}$$

similarly, one defines $B_k, C_k,\ k = 0, 1, 2, \cdots$;

$$\lim_{k \longrightarrow \infty} A_k = \lim_{k \longrightarrow \infty} B_k = \lim_{k \longrightarrow \infty} C_k = \pi/3; \tag{3.8}$$

and $\triangle_1 = \triangle_2 = \cdots = \triangle_k = \cdots = \triangle$.

Theorem 3.2. *In every triangle ABC, if the sequence of* $\{\Delta A_k B_k C_k\}_{k=0}^n$ *that the sides and angles respectively define as* (3.5) *and* (3.6), *then we have*

$$\sum a^2 \geqslant 4\sqrt{3}\Delta + \sum_{k=0}^{n} \left[\sum (a_k - b_k)^2\right] \tag{3.9}$$

Proof. In $\Delta A_{k-1}B_{k-1}C_{k-1}$, utilizing the fact that

$$a_{k-1}^2 = 4\Delta_{k-1}\tan\frac{A_{k-1}}{2} + (b_{k-1}-c_{k-1})^2,$$

we have

$$\sum a_{k-1}^2 = 4\Delta_{k-1}\sum\tan\frac{A_{k-1}}{2} + \sum(b_{k-1}-c_{k-1})^2. \tag{3.10}$$

From (3.6), then (3.10) become

$$\sum a_{k-1}^2 = 4\Delta_{k-1}\sum\cot A_k + \sum(b_{k-1}-c_{k-1})^2. \tag{3.11}$$

Also, by using the laws of cosines, we obtain

$$\sum a_{k-1}^2 = 4\Delta_{k-1}\sum\cot A_{k-1}. \tag{3.12}$$

Combining expansion (3.11) and (3.12), we have

$$4\Delta_{k-1}\sum\cot A_{k-1} = 4\Delta_{k-1}\sum\cot A_k + \sum(b_{k-1}-c_{k-1})^2. \tag{3.13}$$

From $\triangle = \triangle_0 = \triangle_1 = \triangle_2 = \cdots = \triangle_n$, we get

$$4\Delta\sum\cot A_{k-1} = 4\Delta\sum\cot A_k + \sum(b_{k-1}-c_{k-1})^2. \tag{3.14}$$

In (3.14), summing from 1 to $n+1$ for k, we obtain

$$4\Delta\sum\cot A = 4\Delta\sum\cot A_{n+1} + \sum_{k=1}^{n+1}\sum(b_{k-1}-c_{k-1})^2. \tag{3.15}$$

Utilizing the fact that

$$\sum\cot A_{n+1} \geqslant \sqrt{3}$$

and combining expansion (3.15), (3.12) and Lemma3.1, the proof of inequality (3.2) is completed. ∎

4. The Index Generalizations for Weitzenboeck's Type Inequality

Theorem 4.1. *Let $\lambda \geqslant 1$, and $n \in N$, in every ΔABC, we have*

$$\sum a^{2\lambda} \geqslant 4^\lambda 3^{1-\frac{\lambda}{2}}\Delta^\lambda + \sum_{k=0}^{n}\sum|b_k-c_k|^{2\lambda} \tag{4.1}$$

where a_k, b_k, c_k define as Lemma3.1.

Proof. When $\lambda \geqslant 1$, using the fact that [7]

$$\sum a^{2\lambda} \geqslant 3^{1-\lambda}(\sum a^2)^\lambda \tag{4.2}$$

and

$$\left(\sum_{k=1}^{m} x_k\right)^\lambda \geqslant \sum_{k=1}^{m} x_k^\lambda, \tag{4.3}$$

where $x_i \geqslant 0 (i = 1, 2, \cdots, n)$ and combining Theorem3.2, we obtain inequality (4.1). The Theorem4.1 is proved. ∎

To prove the next theorem, the following lemmas [8] are necessary:

Lemma 4.1. *If $k \leqslant (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$, then in every ΔABC we have*

$$\left(\frac{1}{3}\sum a^k\right)^{\frac{1}{k}} \leqslant \sqrt{3}R \tag{4.4}$$

Theorem 4.2. *Let* $0 < \lambda < (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$, *and* $n \in N$, *in every* ΔABC, *we have*

$$\sum \frac{1}{a^{2\lambda}} \leqslant \frac{3^{1+\frac{\lambda}{2}}}{(4\Delta)^{\lambda}} + \frac{1}{2}\sum\left(\frac{1}{a^{\lambda}} - \frac{1}{b^{\lambda}}\right)^2 \tag{4.5}$$

Proof. When $0 < \lambda < (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$, from Lemma4.1, we get

$$\left(\frac{1}{3}\sum a^{\lambda}\right)^{\frac{1}{\lambda}} \leqslant \sqrt{3}R$$

i.e.,

$$\sum a^{\lambda} \leqslant 3^{1+\frac{\lambda}{2}} R^{\lambda}. \tag{4.6}$$

Combining expansion $R = abc/(2\Delta)$, inequality (4.6) becomes (4.5). The Theorem4.2 is proved. ∎

Theorem4.2 is obtained by J. Chen in [13].

Corollary 4.1. *In every* ΔABC, *we have*

$$\sum \frac{1}{a^2} \leqslant \frac{3\sqrt{3}}{4\Delta} + \frac{1}{2}\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2 \tag{4.7}$$

and the coefficient $1/2$ *is the best possible.*

Proof. Let $\lambda = 1$ for (4.5), we easily get inequality (4.7). Let $a = b = 1, c = t$, then inequality $\sum \frac{1}{a^2} \leqslant \frac{3\sqrt{3}}{4\Delta} + \mu\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2$ becomes

$$\frac{1}{t^2}\left[2t^2 + 1 - 2\mu(1-t)^2 - \frac{3\sqrt{3}t}{\sqrt{4-t^2}}\right] \leqslant 0$$

Set $t \to 0$, we can obtain $\mu \geqslant \frac{1}{2}$, therefore the coefficient $\mu = \frac{1}{2}$ is the best possible. ∎

5. Weitzenboeck's Type Inequalities of the Planar Convex Polygon

The next Lemma5.1 is a preliminary election problem of the 29th IMO.

Lemma 5.1. *If* $\alpha_k > 0, \beta_k$ *are real numbers* $(k = 1, 2, \cdots, n)$, *and* $\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = \pi$, *then we have*

$$\sum_{k=1}^{n} \frac{\cos\beta_k}{\sin\alpha_k} \leqslant \sum_{k=1}^{n} \cot\alpha_k \tag{5.1}$$

with equality holding if and only if $\alpha_k = \beta_k$ $(k = 1, 2, \cdots, n)$.

Theorem 5.1. *Assume* a_k $(k = 1, 2, \cdots, n)$ *denote the sides of a planar convex polygon* $\mathrm{A_1A_2\cdots A_n}$ *and* F *the area. If* $\alpha_k > 0$ $(k = 1, 2, \cdots, n)$ *for* $\sum_{k=1}^{n} \alpha_k = \pi$, *then in every planar convex polygon the following inequality holds*

$$\sum_{k=1}^{n} a_k^2 \cot\alpha_k \geqslant 4F \tag{5.2}$$

Proof. To prove planar convex polygon $\mathrm{A_1A_2\cdots A_n}$ inscribed in a circle, because its area is maximal.

Set $\beta_k = 2\gamma_k - \alpha_k$, where $2\gamma_k$ is the angle at the centre of the side a_k, $k = 1, 2, \cdots, n$. From Lemma5.1, we have

$$\sum_{k=1}^{n} \frac{\cos(2\gamma_k - \alpha_k)}{\sin\alpha_k} \leqslant \sum_{k=1}^{n} \cot\alpha_k$$

that is

$$\sum_{k=1}^{n} \sin^2 \gamma_k \cdot \cot \alpha_k \geqslant \sum_{k=1}^{n} \sin 2\gamma_k \tag{5.3}$$

Utilizing the fact that $a_k = 2R \sin \gamma_k$ $(k = 1, 2, \cdots, n)$, and $F = \frac{1}{2} R^2 \sum_{k=1}^{n} \sin 2\gamma_k$, inequalities (5.3) becomes (5.2). Lemma5.1 is proved. ∎

From Theorem5.1, the following Corollary5.1 is obvious.

Corollary 5.1. *Assume a_k $(k = 1, 2, \cdots, n)$ denote the sides of a planar convex polygon* $\mathrm{A_1 A_2 \cdots A_n}$ *and F the area, then in every convex polygon the following inequality holds*

$$\sum_{k=1}^{n} a_k^2 \geqslant 4F \tan \frac{\pi}{n} \tag{5.4}$$

Corollary 5.2. *Assume a_k $(k = 1, 2, 3, 4)$ denote the sides of quadrilateral* ABCD *and F the area. If λ_k $(k = 1, 2, 3, 4)$ are the real numbers for*

$$\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3 > 0,$$

and

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 0,$$

then in every quadrilateral we have

$$\sum_{k=1}^{4} \lambda_k a_k^2 \geqslant 4F \sqrt{\frac{\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}}. \tag{5.5}$$

Proof. Set $t\lambda_k = \cot \alpha_k$ $(k = 1, 2, 3, 4)$, t is a real constant. From Theorem5.1 for $n = 4$, we have

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi,$$

and

$$\tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0.$$

Using the fact that

$$\tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \frac{\displaystyle\sum_{1 \leqslant i < j < k \leqslant 4} \cot \alpha_i \cot \alpha_j \cot \alpha_k - \sum_{k=1}^{4} \cot \alpha_k}{\displaystyle\prod_{k=1}^{4} \cot \alpha_k - \sum_{1 \leqslant j < k \leqslant 4} \cot \alpha_j \cot \alpha_k + 1},$$

we can obtain

$$t^2 = \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3},$$

the proof of Corollary5.2 is completed. ∎

Corollary 5.3. *Assume a_k $(k = 1, 2, 3, 4)$ denote the sides of quadrilateral*ABCD *and F the area, in every quadrilateral we have*

$$\prod_{k=1}^{4} a_k \sum_{k=1}^{4} a_k \sum_{k=1}^{4} a_k^{-1} \geqslant 16F^2 \tag{5.6}$$

Proof. Let $\lambda_1 a_1 = \lambda_2 a_2 = \lambda_3 a_3 = \lambda_4 a_4$ in Corollary5.2, then inequality (5.5) become (5.6). Corollary5.3 is proved. ∎

Remark 5.1. *The results of this section were obtained by X.-Zh. Yang in* [9].

6. The Reverse Weitzenboeck's Inequality

Theorem 6.1. *In every ΔABC, we have*

$$\sum a^2 \leqslant 4\sqrt{3}\Delta + 3\sum (a-b)^2 \tag{6.1}$$

and the coefficient 3 *is the best possible.*

Proof. Utilizing the fact that

$$\prod \sin A = sr/2R^2, \tag{6.2}$$

$$\sum \sin^2 A = (s^2 - 4Rr - r^2)/2R^2, \tag{6.3}$$

and

$$\sum \sin B \sin C = (s^2 + 4Rr + r^2)/4R^2, \tag{6.4}$$

then (6.1)$\Leftrightarrow 2\sqrt{3}\prod \sin A + 5\sum \sin^2 A - 6\sum \sin B \sin C \geqslant 0 \Leftrightarrow \sqrt{3}sr + s^2 - 16Rr - 4r^2 \geqslant 0 \Leftrightarrow$

$$\sqrt{3}r(s - 3\sqrt{3}r) + s^2 - 16Rr + 5r^2 \geqslant 0 \tag{6.5}$$

From Gerretsen's inequalities [10]

$$27r^2 \leqslant 16Rr - 5r^2 \leqslant s^2 \leqslant 4R^2 + 4Rr + 3r^2 \tag{6.6}$$

and Euler inequality $R \geqslant 2r$, inequality (6.5) or (6.1) is true.

Set $a = b = 1, c = t$, then inequality $\sum a^2 \leqslant 4\sqrt{3}\Delta + \mu \sum (b-c)^2$ becomes

$$2 + t^2 \leqslant \sqrt{3}\,t\sqrt{4-t^2} + 2\mu(1-t)^2,$$

Let $t \to 2$, we obtain $\mu \geqslant 3$, therefore the coefficient $\mu = 3$ is the best possible. The proof of Theorem6.1 is completed. ∎

Theorem 6.2. *In every ΔABC, we have*

$$\sum a^2 \leqslant 4\sqrt{3}\Delta + \frac{3}{2}\sum (a-b)^2 + 2R^2 \sum (\cos B - \cos C)^2 \tag{6.7}$$

Proof. Utilizing the fact that (6.2) ,(6.3), (6.4) and

$$\sum \cos B \cos C = (s^2 - 4R^2 + r^2)/4R^2, \tag{6.8}$$

then

$$(6.7) \Leftrightarrow 2\sqrt{3}\prod \sin A + \sum \sin^2 A - 3\sum \sin B \sin C - \sum \cos B \cos C + 3 \geqslant 0$$
$$\Leftrightarrow 2\sqrt{3}sr - s^2 + 8R^2 - 10Rr - 3r^2 \geqslant 0 \Leftrightarrow$$

$$2\sqrt{3}r(s - 3\sqrt{3}r) + 2(R-2r)(2R-3r) + 4R^2 + 4Rr + 3r^2 - s^2 \geqslant 0 \tag{6.9}$$

From Gerretsen's inequalities (6.6) and Euler inequality $R \geqslant 2r$, inequality (6.9) or (6.7) is proved. ∎

Above two results are obtained by B.-Q. Liu in [11].

The next Lemma6.10 is proved by A.Oppenheim in [12]:

Lemma 6.1. *If $0 < \theta \leqslant 1$, then in every ΔABC we have*

$$(\sum a^\theta) \prod (b^\theta + c^\theta - a^\theta) \geqslant 3^{1-\theta}(4\Delta)^{2\theta} \tag{6.10}$$

Theorem 6.3. *If $\lambda \geqslant 2$, in every ΔABC we have*

$$\sum a^{2\lambda} \leqslant 4^\lambda 3^{1-\frac{\lambda}{2}}\Delta^\lambda + \sum (b^\lambda - c^\lambda)^2 \tag{6.11}$$

Proof. Since (6.11) is equivalence with

$$(\sum a^{\frac{\lambda}{2}})\prod(b^{\frac{\lambda}{2}}+c^{\frac{\lambda}{2}}-a^{\frac{\lambda}{2}})\leqslant 4^{\lambda}3^{1-\frac{\lambda}{2}}\Delta^{\lambda} \tag{6.12}$$

If $a_1^{\frac{\lambda}{2}}, a_2^{\frac{\lambda}{2}}, a_3^{\frac{\lambda}{2}}$ are not three sides of a triangle, then the expansion (6.12) or (6.11) is true, and if $a_1^{\frac{\lambda}{2}}, a_2^{\frac{\lambda}{2}}, a_3^{\frac{\lambda}{2}}$ are three sides of a triangle, from Lemma6.1 and $0 < 2/\lambda \leqslant 1$, then we can obtain the expansion (6.12) or (6.11). Theorem6.3 is proved. ∎

Finally, we propose an open question:

Assume $\lambda > (\ln 9 - \ln 4)/(\ln 4 - \ln 3)$. Solve that the best possible μ for the following inequality holds

$$\sum \frac{1}{a^{2\lambda}} \leqslant \frac{3^{1+\frac{\lambda}{2}}}{(4\Delta)^{\lambda}} + \mu \sum \left(\frac{1}{a^{\lambda}} - \frac{1}{b^{\lambda}}\right)^2. \tag{6.13}$$

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