INEQUALITIES CHECKED BY FUNCTIONS OF BOUNDED VARIATION

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Abstract. In this article we will prove some inequalities checked by functions of bounded variations and then we will give applications of this inequalities.

1. Introduction

Remind some known results in connection with the functions of bounded variation.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$, f be a derivable function on $[a,b]$, with the derivative f' integrable Riemann on $[a,b]$. Then the function f is of bounded variation on $[a,b]$ and*

$$\bigvee_a^b (f) = \int_a^b |f'(x)|\, dx \tag{1.1}$$

where $\bigvee_a^b(f)$ is the total variation of the function f on $[a,b]$.

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is monotone function, then*

$$\bigvee_a^b (f) = |f(b)-f(a)|\,. \tag{1.2}$$

For the demonstration of the Theorem 1, the Lemma 1 or other characterization of the functions of bounded variation, we can consult [2].

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$, be a derivable function on $[a,b]$ with the derivative f' continuous on $[a,b], n\in\mathbb{N}^*, a=a_0<a_1<...<a_{n+1}=b$, f' has change of sign only in the points $a_1,a_2,...,a_n$. Then*

$$\bigvee_a^b (f) = \begin{cases} \left| f(b)-f(a)+2\sum\limits_{k=1}^{n}(-1)^{k-1}f(a_k)\right|, & \text{if } n \text{ is even} \\ \left| -f(b)-f(a)+2\sum\limits_{k=1}^{n}(-1)^{k-1}f(a_k)\right|, & \text{if } n \text{ is odd} \end{cases} \tag{1.3}$$

Proof. In addition to Theorem 1 we have

$$\bigvee_a^b (f) = \int_a^b |f'(x)|\, dx = \sum_{k=0}^{n}\int_{a_k}^{a_{k+1}} |f'(x)|\, dx = \left|\sum_{k=0}^{n}(-1)^k \int_{a_k}^{a_{k+1}} f'(x)dx\right|$$

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$$= \left|\sum_{k=0}^{n}(-1)^k f(x)\Big|_{a_k}^{a_{k+1}}\right|$$
$$= |(f(a_1)-f(a_0))-(f(a_2-f(a_1))+...+(-1)^n(f(a_{n+1})-f(a_n)|\,,$$

out of which (1.3) results. ∎

2. **Main Results**

Theorem 2. *If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on $[a,b]$, then*

$$\left|\int_a^{\frac{a+b}{2}} f(x)dx-\int_{\frac{a+b}{2}}^b f(x)dx+t[f(b)-f(a)](b-a)\right| \le \frac{b-a}{2}\bigvee_a^b(f)\cdot\cdot\max\{2t,|2t-1|\} \tag{2.1}$$

for all $t\in[0,1]$.

Proof. Let $s_t:[a,b]\to\mathbb{R}, t\in[0,1]$ be a function defined by

$$s_t(x)=\begin{cases}[(1-t)a+tb]-x, & x\in\left[a,\frac{a+b}{2}\right)\\ x-[ta+(1-t)b], & x\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Since

$$s_t\left(\frac{a+b}{2}-0\right)=s_t\left(\frac{a+b}{2}+0\right)=s_t\left(\frac{a+b}{2}\right)=\frac{(2t-1)(b-a)}{2},\qquad t\in[0,1]$$

and heeding definition of the function $s_t, t\in[0,1]$, results that the function s_t is continuous on $[a,b]$, $t\in[0,1]$. Knowing that the function s_t is continuous on $[a,b]$ and the function f is of bounded variation, we have that the function s_t is integrable Stieltjes-Riemann reported to the function f. Then

$$\int_a^b s_t(x)df(x)=\int_a^{\frac{a+b}{2}} s_t(x)df(x)++\int_{\frac{a+b}{2}}^b s_t(x)df(x)$$

and applying the integration through parts formula for the integral Stieltjes-Riemann, we have

$$\begin{aligned}\int_a^b s_t(x)df(x) &= \left[s_t(x)f(x)\Big|_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}} s_t'(x)f(x)dx\right]\\ &\quad+\left[s_t(x)f(x)\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b s_t'(x)f(x)dx\right]\\ &=\frac{(2t-1)(b-a)}{2}f\left(\frac{a+b}{2}\right)-t(b-a)f(a)+t(b-a)f(b)\\ &\quad-\frac{(2t-1)(b-a)}{2}f\left(\frac{a+b}{2}\right)+\int_a^{\frac{a+b}{2}} f(x)dx-\int_{\frac{a+b}{2}}^b f(x)dx,\end{aligned}$$

from where

$$\int_a^b s_t(x)df(x)=\int_a^{\frac{a+b}{2}} f(x)dx-\int_{\frac{a+b}{2}}^b f(x)dx+t(b-a)[f(b)-f(a)], \tag{2.2}$$

for all $t \in [0,1]$.Let there be the sequence of divisions $(\Delta_n)_{n\geq 1}, \Delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{k_n}^{(n)} = b$, with $\lim_{n\to\infty} \parallel \Delta_n \parallel = 0$, where $\parallel \Delta_n \parallel = \max_{i=\overline{0,k_n-1}} (x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$, $i = \overline{0, k_n - 1}$. Because s_t is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, we have that

$$\int_a^b s_t(x)df(x) = \lim_{n\to\infty} \sum_{i=0}^{k_n-1} s_t\left(\xi_i^{(n)}\right)\left[f\left(x_{i+1}^{(n)}\right) - f\left(x_i^{(n)}\right)\right]$$

from where

$$\begin{aligned}\left|\int_a^b s_t(x)df(x)\right| &= \left|\lim_{n\to\infty} \sum_{i=0}^{k_n-1} s_t\left(\xi_i^{(n)}\right)\left[f\left(x_{i+1}^{(n)}\right) - f\left(x_i^{(n)}\right)\right]\right| \\ &\leq \lim_{n\to\infty} \sum_{i=0}^{k_n-1} \left|s_t\left(\xi_i^{(n)}\right)\right|\left|f\left(x_{i+1}^{(n)}\right) - f\left(x_i^{(n)}\right)\right| \\ &\leq \max_{x\in[a,b]} |s_t(x)| \lim_{n\to\infty} \sum_{i=0}^{k_n-1} \left|f\left(x_{i+1}^{(n)}\right) - f\left(x_i^{(n)}\right)\right|,\end{aligned}$$

from where

$$\left|\int_a^b s_t(x)df(x)\right| \leq \max_{x\in[a,b]} |s_t(x)| \bigvee_a^b (f). \tag{2.3}$$

Considering that s_t is strictly decreasing on $[a, \frac{a+b}{2})$, strictly increasing on $[\frac{a+b}{2}, b]$ and that

$$s_t(a) = s_t(b) = t(b-a),$$
$$s_t\left(\frac{a+b}{2}\right) = \frac{(2t-1)(b-a)}{2},$$

we have that

$$\max_{x\in[a,b]} |s_t(x)| = \frac{b-a}{2} \max\{2t, |2t-1|\} \tag{2.4}$$

for all $t \in [0,1]$.

From the relations (2.2)–(2.4) the inequality (2.1) follows. ∎

Corollary 1. *If $f : [a,b] \to \mathbb{R}$ is of bounded variation function on $[a,b]$ then*

$$\left|\int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx\right| \leq \frac{b-a}{2} \bigvee_a^b (f), \tag{2.5}$$

$$\left|\int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx + \frac{f(b)-f(a)}{2}(b-a)\right| \leq \frac{b-a}{2} \bigvee_a^b (f) \tag{2.6}$$

and

$$\left|\int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx + [f(b)-f(a)](b-a)\right| \leq (b-a) \bigvee_a^b (f). \tag{2.7}$$

Proof. In the inequality (2.1) we take $t = 0, t = \frac{1}{2}$ and $t = 1$ respectively. ∎

Corollary 2. *If $f : [a, b] \to \mathbb{R}$ is of bounded variation function on $[a, b]$, then*

$$\left| \int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx + t[f(b) - f(a)](b-a) \right| \le \frac{b-a}{2} \bigvee_a^b (f) \cdot \max\{2t, |2t-1|\} \le (b-a) \bigvee_a^b (f) \tag{2.8}$$

for all $t \in [0, 1]$.

Proof. Consider the function $g : [0, 1] \to \mathbb{R}$, $g(t) = 2t - |2t-1|$, $t \in [0, 1]$. We have

$$g(t) = \begin{cases} 4t - 1, & t \in \left[0, \frac{1}{2}\right), \\ 1, & t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

from where

$$\max\{2t, |2t-1|\} = \begin{cases} 1 - 2t, & t \in \left[0, \frac{1}{4}\right), \\ 2t, & t \in \left[\frac{1}{4}, 1\right], \end{cases}$$

and $\max\limits_{t \in [0,1]} \max\{2t, |2t-1|\} = 2$. Using (2.1), the inequalities (2.8) are obtained. ∎

Corollary 3. *If $f : [a, b] \to \mathbb{R}$ is of bounded variation function on $[a, b]$ then*

$$\max\left\{ \left| \int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx \right|, \left| \int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx + [f(b) - f(a)](b-a) \right| \right\} \le (b-a) \bigvee_a^b (f). \tag{2.9}$$

Proof. The function $h : [0, 1] \to \mathbb{R}$,

$$h(t) = t[f(b) - f(a)](b-a) + \int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx,$$

is of maximum degree 1, so the extreme values of the function are reached for $t = 0$ and $t = 1$. The Corollary 2 is applied next. ∎

Corollary 4. *If $f : [a, b] \to \mathbb{R}$ is a monotone function, then*

$$\begin{aligned} -2(b-a)[f(b) - f(a)] &\le -\frac{b-a}{2}[f(b) - f(a)][\max\{2t, |2t-1|\} + 2t] \\ &\le \int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx \\ &\le \frac{b-a}{2}[f(b) - f(a)][\max\{2t, |2t-1|\} - 2t] \\ &\le \frac{b-a}{2}[f(b) - f(a)] \end{aligned} \tag{2.10}$$

for all $t \in [0, 1]$.

Proof. Because f is a monotone function, according to Lemma 1, we have that $\bigvee_a^b(f) = |f(b) - f(a)|$. Applying the Theorem 2, we have

$$\begin{aligned} &-\frac{b-a}{2}[f(b) - f(a)][\max\{2t, |2t-1|\} + 2t] \\ &\leq \int_a^{\frac{a+b}{2}} f(x)dx - \int_{\frac{a+b}{2}}^b f(x)dx \\ &\leq \frac{b-a}{2}[f(b) - f(a)][\max\{2t, |2t-1|\} - 2t] \end{aligned} \tag{2.11}$$

for all $t \in [0, 1]$.

Heeding the demonstration of the Corollary 2, we have

$$\max\{2t, |2t-1|\} + 2t = \begin{cases} 1, & t \in \left[0, \frac{1}{4}\right), \\ 4t, & t \in \left[\frac{1}{4}, 1\right], \end{cases}$$

$$\max\{2t, |2t-1|\} - 2t = \begin{cases} 1 - 4t, & t \in \left[0, \frac{1}{4}\right), \\ 0, & t \in \left[\frac{1}{4}, 1\right], \end{cases}$$

from where

$$\max_{t\in[0,1]}[\max\{2t, \mid 2t-1 \mid\} + 2t] = 4 \tag{2.12}$$

and

$$\max_{t\in[0,1]}[\max\{2t, \mid 2t-1 \mid\} - 2t] = 1. \tag{2.13}$$

From (2.11)–(2.13) we get the relation (2.10). ∎

Next, we will give some applications of the inequalities demonstrated in this article, where we will make use of the Theorem 1 and the Lemma 1.

Application 2.1. If $0 < a < \frac{\pi}{2}$, then

$$|\cos a - 1 + 2at \sin a| \leq a \sin a : \max\{2t, |2t-1|\} \leq 2a \sin a \tag{2.14}$$

for all $t \in [0, 1]$.

Proof. Consider the function $f : [-a, a] \to \mathbb{R}, f(x) = \sin x$ and apply the inequality (2.8). ∎

Application 2.2. If $a \leq b$, then

$$\begin{aligned} \max\left\{\left|2e^{\frac{a+b}{2}} - e^a - e^b\right|, \left|2e^{\frac{a+b}{2}} - e^a - e^b + (b-a)(e^b - e^a)\right|\right\} \\ \leq (b-a)\left(e^b - e^a\right). \end{aligned} \tag{2.15}$$

Proof. Consider the function $f : [a, b] \to \mathbb{R}, f(x) = e^x$ and apply the inequality (2.9). ∎

Application 2.3. If $0 < a \leq b$, then

$$\left(\frac{b}{a}\right)^{-2(b-a)} \leq \left(\frac{a+b}{2}\right)^{a+b} \frac{1}{a^a b^b} \leq \left(\frac{b}{a}\right)^{\frac{b-a}{2}}. \tag{2.16}$$

Proof. Consider the function $f : [a, b] \to \mathbb{R}, f(x) = \ln x$ and apply the inequality (2.10). ∎

Application 2.4. If $a < 0 < b$ and $n \in \mathbb{N}^*$, then

$$\left|\frac{2}{2n+1}\left[\left(\frac{a+b}{2}\right)^{2n+1} - \frac{a^{2n+1}+b^{2n+1}}{2}\right] - t(b^{2n}-a^{2n})(b-a)\right| \tag{2.17}$$
$$\le \frac{b-a}{2}\left(a^{2n}+b^{2n}\right)\max\{2t, |2t-1|\} \le (b-a)(a^{2n}+b^{2n})$$

for all $t \in [0,1]$.

Proof. Consider the function $f : [a,b] \to \mathbb{R}, f(x) = x^{2n}$. Then Theorem 2. and Theorem 1 are taken aware of. ∎

Application 2.5. If $0 < a < \frac{\pi}{2}$ and $\frac{3\pi}{2} < b < 2\pi$, then

$$\max\left\{\left|\cos a + \cos b - 2\cos\frac{a+b}{2}\right|,\right. \tag{2.18}$$
$$\left.\left|\cos a + \cos b - 2\cos\frac{a+b}{2} + (b-a)(\sin b - \sin a)\right|\right\}$$
$$\le (b-a)(4 - \sin a + \sin b).$$

Proof. Consider the function $f : [a,b] \to \mathbb{R}, f(x) = \sin x$. Then using (2.9), the Theorem 1 or the Lemma 2, the desired result follows. ∎

References

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