THE GENERALIZED WILKINS' INEQUALITY

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ABSTRACT. In this paper, the generalization of Wilkins' inequality is deduced, and some known results are obtained.

1. Introduction

In 1969, O.Bottema [1] noted the following Wilkins' triangle inequality:

Theorem 1.1. *If A, B, C be the angles of triangle ABC, then*

$$\sin A \sin B \sin \frac{C}{2} \le \frac{2\sqrt{3}}{9} \tag{1.1}$$

with equality holding if and only if triangle ABC is a regular triangle.

In this paper, we give a generalization of Wilkins' triangle inequality (1.1), and by its application, some known results are obtained.

2. Main Results

Theorem 2.1. *Let $m_i \geq 1, x_i > 0, \alpha_i \in (0, \pi)$ $(i = 1, 2, \ldots, n, n \geqslant 2)$, and $\sum_{i=1}^n \alpha_i = \theta \leq \pi$, then*

$$\sum_{i=1}^n x_i \left(\sqrt{1 + \frac{m_i^2}{\lambda^2 x_i^2}} \cdot \sin \frac{\alpha_i}{m_i} \right)^k \le \sum_{i=1}^n x_i \tag{2.1}$$

if $0 < k \leq 1$, and the reverse inequality holds if $k < 0$. With equality holding if and only if $\lambda = \frac{m_i}{x_i} \tan \frac{\alpha_i}{m_i} (i = 1, 2, \cdots, n)$, where λ is a positive root of the following equation

$$\sum_{i=1}^n m_i \arctan \frac{\lambda x_i}{m_i} = \theta \tag{2.2}$$

Proof. Set a function

$$f(x) = \sum_{i=1}^n m_i \arctan \frac{\lambda x_i}{m_i} - \theta. \tag{2.3}$$

It can easily be seen that f is a continuous and monotone increasing function in the interval $[0, +\infty)$.

Because $f(0) = -\theta < 0$, $\lim_{\lambda \to +\infty} f(x) = \frac{\pi}{2} \sum_{i=1}^n m_i - \theta > 0$, therefore the equation (2.2) can only hold for positive roots.

Let $\beta_i = m_i \arctan \frac{\lambda x_i}{m_i}$, then $\lambda = \frac{m_i}{x_i} \tan \frac{\beta_i}{m_i}$, and $\beta_i > 0 (i = 1, 2, \ldots, n)$.

From (2.2), we have

$$\sum_{i=1}^n \beta_i = \theta.$$

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Set $x, x_0 \in (0, \pi)$, by using Taylor's formula, we have

$$\sin x = \sin x_0 + (x - x_0)\cos x_0 - \frac{1}{2}(x - x_0)^2 \sin \xi,$$

where ξ is between x_0 and x.

Pay attention to $\sin x_0 > 0, \sin \xi > 0$, then

$$\sin x \leq \sin x_0 + (x - x_0)\cos x_0$$

or

$$\frac{\sin x}{\sin x_0} \leq 1 + \frac{x - x_0}{\tan x_0}$$

with equality holding if only if $x = x_0$.

Let $x = \frac{\alpha_i}{m_i}, x_0 = \frac{\beta_i}{m_i}$, we have

$$\frac{\sin \frac{\alpha_i}{m_i}}{\sin \frac{\beta_i}{m_i}} \leq 1 + \frac{\alpha_i - \beta_i}{m_i \tan \frac{\beta_i}{m_i}} = 1 + \frac{\alpha_i - \beta_i}{\lambda x_i}$$

If $0 < k \leq 1$, we obtain

$$\left(\frac{\sin \frac{\alpha_i}{m_i}}{\sin \frac{\beta_i}{m_i}}\right)^k \leq \left(1 + \frac{\alpha_i - \beta_i}{\lambda x_i}\right)^k.$$

Let $i = 1, 2, \ldots, n$, by using the weighted Arithmetic-Geometric mean inequality, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = \theta$, we have

$$\sum_{i=1}^n x_i \left(\frac{\sin \frac{\alpha_i}{m_i}}{\sin \frac{\beta_i}{m_i}}\right)^k \leq \sum_{i=1}^n x_i \left(1 + \frac{\alpha_i - \beta_i}{\lambda x_i}\right)^k$$

$$\leq \sum_{i=1}^n x_i \left[\frac{\sum_{i=1}^n x_i \left(1 + \frac{\alpha_i - \beta_i}{\lambda x_i}\right)}{\sum_{i=1}^n x_i}\right]^k = \sum_{i=1}^n x_i$$

from

$$\sin \frac{\beta_i}{m_i} = \left(1 + \cot^2 \frac{\beta_i}{m_i}\right)^{-1/2} = \left(1 + \frac{m_i^2}{\lambda x_i^2}\right)^{-\frac{1}{2}},$$

and if $0 < k \leq 1$, the proof of inequality (2.1) is completed. By all appearances, the reverse inequality holds if $k < 0$. With equality holding if and only if $\alpha_i = \beta_i = m_i \arctan \frac{\lambda x_i}{m_i}$, or

$$\lambda = \frac{m_i}{x_i} \tan \frac{\alpha_i}{m_i} \quad (i = 1, 2, \ldots, n).$$

Thus, the proof of Theorem 2.1 is complete. ∎

Theorem 2.2. *Let $m_i \geq 1, x_i > 0, \alpha_i \in (0, \pi)$ $(i = 1, 2, \ldots, n)$, and $\sum_{i=1}^n \alpha_i = \theta \leq \pi$, then*

$$\prod_{i=1}^n \sin^{x_i} \frac{\alpha_i}{m_i} \leq \prod_{i=1}^n \left(1 + \frac{m_i^2}{\lambda^2 x_i^2}\right)^{\frac{-x_i}{2}} \tag{2.4}$$

With equality holding if and only if $\lambda = \frac{m_i}{x_i} tg \frac{\alpha_i}{m_i}$ $(i = 1, 2, \ldots, n)$, where λ is a positive root of equation (2.2).

Proof. From Theorem 2.1,

$$\lim_{r \to 0} \left(\sum_{i=1}^n p_i a_i^r\right)^{1/r} = \prod_{i=1}^n a_i^{p_i} \quad \left(\text{where} \quad \sum_{i=1}^n p_i = 1\right)$$

and using standard arguments, the proof of Theroem 2.2 is complete. ∎

3. Some Particular Triangle Inequalities

The following proposition holds

Proposition 3.1. *Let* $m_i > 0, \alpha_i > 0$ $(i = 1, 2, \ldots, n)$*, and* $\sum_{i=1}^{n} \alpha_i = \theta \leq \pi$*, then*

$$\sum_{i=1}^{n} \sin^k \frac{\alpha_i}{m_i} \leq \sum_{i=1}^{n} m_i \sin^k \left(\frac{\theta}{\sum_{i=1}^{n} m_i} \right), \tag{3.1}$$

if $0 < k \leq 1$*, and*

$$\prod_{i=1}^{n} \sin^{m_i} \frac{\alpha_i}{m_i} \leq \sin^{\sum_{i=1}^{n} m_i} \left(\frac{\theta}{\sum_{i=1}^{n} m_i} \right). \tag{3.2}$$

With both equalities holding if and only if $\alpha_1 : m_1 = \alpha_2 : m_2 = \cdots = \alpha_n : m_n$.

Proof. Let $x_i = m_i \geq 1$, from Theorem 2.1 and Theorem 2.2, we have $\lambda = \tan \left(\frac{\theta}{\sum_{i=1}^{n} m_i} \right)$, and the inequalities (3.1) and (3.2). ∎

Proposition 3.2. *Let* $x_i, \alpha_i \in \mathbb{R}^+$ $(i = 1, 2, 3, 4)$*, and* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$*, then*

$$\begin{aligned} &\sqrt{(x_1+x_2)(x_1+x_3)(x_1+x_4)} \sin \alpha_1 + \sqrt{(x_2+x_1)(x_2+x_3)(x_2+x_4)} \sin \alpha_2 \\ &\quad + \sqrt{(x_3+x_1)(x_3+x_2)(x_3+x_4)} \sin \alpha_3 + \sqrt{(x_4+x_1)(x_4+x_2)(x_4+x_3)} \sin \alpha_4 \\ &\qquad \leq \left(\sum_{i=1}^{n} x_i \right)^{\frac{3}{2}} \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\frac{x_1^2}{\sqrt{(x_1+x_2)(x_1+x_3)(x_1+x_4)} \sin \alpha_1} + \frac{x_2^2}{\sqrt{(x_2+x_1)(x_2+x_3)(x_2+x_4)} \sin \alpha_2} \\ &\quad + \frac{x_3^2}{\sqrt{(x_3+x_1)(x_3+x_2)(x_3+x_4)} \sin \alpha_3} + \frac{x_4^2}{\sqrt{(x_4+x_1)(x_4+x_2)(x_4+x_3)} \sin \alpha_4} \\ &\qquad \leq \left(\sum_{i=1}^{n} x_i \right)^{\frac{1}{2}} \end{aligned} \tag{3.4}$$

$$\prod_{i=1}^{4} \sin^{x_i} \alpha_i \leq \prod_{i=1}^{4} x_i^{x_i} \sqrt{\left(\sum_{i=1}^{4} x_i \right)^{\sum_{i=1}^{4} x_i} \Bigg/ \prod_{1 \leq i \prec j \leq 4} (x_i + x_j)^{x_i + x_j}} \tag{3.5}$$

with several equalities holding if and only if $x_1 : x_2 : x_3 : x_4 = \tan \alpha_1 : \tan \alpha_2 : \tan \alpha_3 : \tan \alpha_4$.

Proof. Let $m_1 = m_2 = m_3 = m_4 = 1$, $n = 4, \theta = \pi$, $k = 1$ and $k = -1$, then

$$\lambda = \sqrt{\frac{x_1 + x_2 + x_3 + x_4}{(x_1 + x_2) x_3 x_4 + x_1 x_2 (x_3 + x_4)}}$$

and the standard arguments produce there inequality of Proposition3.2, with their equalities holding if and only if $\lambda = \frac{\tan \alpha_i}{x_i} (i = 1, 2, 3, 4)$, i.e.$x_1 : x_2 : x_3 : x_4 = \tan \alpha_1 : \tan \alpha_2 : \tan \alpha_3 : \tan \alpha_4$.The proof of Proposition3.2 is completed. ∎

Proposition 3.3. *If* $x, y, z, t > 0$*, and* $\alpha_i \in R^+ (i = 1, 2, 3, 4)$*,* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$*, then*

$$x \sin \alpha_1 + y \sin \alpha_2 + z \sin \alpha_3 + t \sin \alpha_4 \leq \left[\frac{(xy + zt)(xz + yt)(xt + yz)}{xyzt} \right]^{1/2} \tag{3.6}$$

with equality holding if and only if $x \cos \alpha_1 = y \cos \alpha_2 = z \cos \alpha_3 = t \cos \alpha_4$.

The inequality (3.6) is obtained by Xue-Zhi Yang in 1992 (see [2]).

Proof. Let

$$\begin{aligned} x &= \sqrt{(x_1+x_2)(x_1+x_3)(x_1+x_4)}, \\ y &= \sqrt{(x_2+x_1)(x_2+x_3)(x_1+x_4)}, \\ z &= \sqrt{(x_3+x_1)(x_3+x_2)(x_3+x_4)}, \\ t &= \sqrt{(x_4+x_1)(x_4+x_2)(x_4+x_3)} \end{aligned}$$

then

$$xy+zt = (x_1+x_2+x_3+x_4)\sqrt{(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)}.$$

We similarly define $xz+yt$ and $xt+yz$ to obtain

$$(xy+zt)(xz+yt)(xt+yz) = (x_1+x_2+x_3+x_4)^3 xyzt$$

From inequality (3.4), we have inequality (3.6), with equality holding if and only if $x_i = u\cdot\tan\alpha_i (i = 1,2,3,4)$. From Proposition 3.2, we obtain

$$\begin{aligned} x &= \sqrt{(x_1+x_2)(x_1+x_3)(x_1+x_4)} \\ &= \sqrt{u^3(\tan\alpha_1+\tan\alpha_2)(\tan\alpha_1+\tan\alpha_3)(\tan\alpha_1+\tan\alpha_4)} \\ &= \sqrt{u^3\cdot\frac{\sin(\alpha_1+\alpha_2)\cdot\sin(\alpha_1+\alpha_3)\cdot\sin(\alpha_1+\alpha_4)}{\cos^3\alpha_1\cos\alpha_2\cos\alpha_3\cos\alpha_4}}. \end{aligned}$$

Rearranging we obtain

$$x\cos\alpha_1 = \sqrt{u^3\cdot\frac{\sin(\alpha_1+\alpha_2)\cdot\sin(\alpha_1+\alpha_3)\cdot\sin(\alpha_1+\alpha_4)}{\cos\alpha_1\cos\alpha_2\cos\alpha_3\cos\alpha_4}}.$$

Similarly we have

$$y\cos\alpha_2 = \sqrt{u^3\cdot\frac{\sin(\alpha_1+\alpha_2)\cdot\sin(\alpha_2+\alpha_3)\cdot\sin(\alpha_2+\alpha_4)}{\cos\alpha_1\cos\alpha_2\cos\alpha_3\cos\alpha_4}}$$

and another two formulas for z and t. Pay attention to $\alpha_1+\alpha_2+\alpha_3+\alpha_4=\pi$, we have

$$\begin{aligned} \sin(\alpha_1+\alpha_2) &= \sin(\alpha_3+\alpha_4), \\ \sin(\alpha_1+\alpha_3) &= \sin(\alpha_2+\alpha_4), \\ \sin(\alpha_1+\alpha_4) &= \sin(\alpha_2+\alpha_3) \end{aligned}$$

Thus, the inequality (3.6) holds with equality holding if and only if

$$x\cos\alpha_1 = y\cos\alpha_2 = z\cos\alpha_3 = t\cos\alpha_4.$$

This completes the proof. ∎

Proposition 3.4. *Let $x,y,z>0$, and in every triangle we have the inequalities*

$$\sqrt{\frac{x}{y+z}}\sin A + \sqrt{\frac{y}{z+x}}\sin B + \sqrt{\frac{z}{x+y}}\sin C \le \sqrt{\frac{(x+y+z)^3}{(x+y)(y+z)(z+x)}} \tag{3.7}$$

$$\sin^x A\cdot\sin^y B\cdot\sin^z C \le \sqrt{\frac{x^x y^y x^z (x+y+z)^{x+y+z}}{(x+y)^{x+y}(y+z)^{y+z}(z+x)^{z+x}}} \tag{3.8}$$

with both equalities holding if and only if $x : y : z = \tan A : \tan B : \tan C$ *or*

$$\frac{\sin^2 A}{x(y+z)} = \frac{\sin^2 B}{y(z+x)} = \frac{\sin^2 C}{z(x+y)}.$$

The inequalities (3.7) and (3.8) were obtained by Ke-Chang Yang in 1990 (see [3]).

Proof. Let $x_4 = \alpha_4 = 0$ or $n = 3, m_1 = m_2 = m_3 = 1$. Proposition 3.4 follows from Proposition 3.2 or Theorem 2.1 with both equalities holding if and only if $x : y : z = \tan A : \tan B : \tan C$, or

$$\frac{\sin^2 A}{x(y+z)} = \frac{\sin^2 B}{y(z+x)} = \frac{\sin^2 C}{z(x+y)}$$

because

$$\tan A = \frac{2\sin A \sin B \sin C}{(\sin^2 B + \sin^2 C - \sin^2 A)}.$$

∎

Proposition 3.5. *If* $k, u, v, w > 0$, *and*

$$\frac{1}{u^2+k} + \frac{1}{v^2+k} + \frac{1}{w^2+k} = \frac{2}{k} \tag{3.9}$$

in every triangle, we have the inequality

$$u\sin A + v\sin B + w\sin C \le \frac{1}{k}\sqrt{(u^2+k)(v^2+k)(w^2+k)} \tag{3.10}$$

with equality holding if and only if

$$\frac{u^2+k}{u}\sin A = \frac{v^2+k}{v}\sin B = \frac{w^2+k}{w}\sin C$$

or

$$u\cos A = v\cos B = w\cos C.$$

Proposition 3.5 was obtained by Ke-Chang Yang in 1987 (see [4]).

Proof. Let

$$u = \sqrt{\frac{kx}{y+z}}, \quad v = \sqrt{\frac{ky}{z+x}}, \quad \text{and} \quad w = \sqrt{\frac{kz}{x+y}}.$$

It is easy obtain (3.9), and from inequality (3.7), we have (3.10), with equality holding if and only if

$$\frac{u^2+k}{u}\sin A = \frac{v^2+k}{v}\sin B = \frac{w^2+k}{w}\sin C$$

or

$$u\cos A = v\cos B = w\cos C$$

This completes the proof. ∎

The proofs of the following propositions will be left to the readers.

Proposition 3.6. *Let* $x, y, z > 0$, *in every triangle we have the inequalities*

$$\frac{\sin\frac{A}{2}}{\sqrt{y+z}} + \frac{\sin\frac{B}{2}}{\sqrt{z+x}} + \frac{\sin\frac{C}{2}}{\sqrt{x+y}} \le \frac{x+y+z}{\sqrt{(x+y)(y+z)(z+x)}}$$

and

$$\sin^x\frac{A}{2}\cdot\sin^y\frac{B}{2}\cdot\sin^z\frac{C}{2} \le \frac{x^x y^y z^z}{\sqrt{(x+y)^{x+y}(y+z)^{y+z}(z+x)^{z+x}}}$$

with both equalities holding if and only if $x : y : z = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2}$.

Proposition 3.7. *Let $\lambda_1, \lambda_2, \lambda_3 > 0$, in every triangle we have the inequality*

$$2\lambda_2\lambda_3 \cos\frac{A}{2} + 2\lambda_3\lambda_1 \cos\frac{B}{2} + 2\lambda_1\lambda_2 \cos\frac{C}{2} \le \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

with equality holding if and only if $\lambda_1 : \lambda_2 : \lambda_3 = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$.

Proposition 3.8. *Let $x, y, z > 0$, in every triangle we have the inequalities*

$$\sqrt{\frac{2x+z}{2y+z}}\sin A + \sqrt{\frac{2y+z}{2x+z}}\sin B + 2\sqrt{\frac{z}{x+y}}\sin\frac{C}{2} \le \sqrt{\frac{(x+y+z)^3}{(x+y)(2x+z)(2y+z)}}$$

and

$$\sin^x A \cdot \sin^y B \cdot \sin^z \frac{C}{2} \le \sqrt{\frac{4^{x+y}x^{2x}y^{2y}x^z(x+y+z)^{x+y+z}}{(x+y)^{x+y}(2x+z)^{2x+z}(2y+z)^{2y+z}}}$$

with equality holding if and only if $x : y : z = \tan A : \tan B : 2\tan\frac{C}{2}$.

Proposition 3.9. *Let $m \ge 1, u > 0$, in every triangle we have the inequalities*

$$\sqrt{1+v^2}\sin\frac{A}{m} + \sqrt{1+v^2}\sin\frac{B}{m} + u\sqrt{1+4v^2}\sin\frac{C}{2m} \le 2+u$$

and

$$\sin\frac{A}{m}\cdot\sin\frac{B}{m}\cdot\sin^u\frac{C}{2m} \le (1+v^2)(1+4v^2)^{-u/2},$$

with equality holding if and only if $A = B = m\arctan\frac{1}{v}$, where

$$v = \frac{1}{4}\left[(2+u)\cot\frac{\pi}{2m} + \sqrt{(2+u)^2\cot^2\frac{\pi}{2m} + 8u}\right].$$

Proposition 3.10. *Let $u > 0$, in every triangle we have the inequality*

$$\sin A + \sin B + u\cdot\sin C \le 2(1-v^2)^{2/3}(1-2v^2)^{-1}$$

with equality holding if and only if $A = B = \arccos v$, where $v = 2u(1+\sqrt{1+8u^2})^{-1}$.

4. Triangle Inequalities for the Sides and Area

Let a, b, c be the sides of a triangle ABC, and S the area, then the following proposition holds

Proposition 4.1. *Let $x, y, z > 0$, in every triangle we have the inequality*

$$(2S)^{x+y+z} \le \sqrt{\frac{x^x y^y x^z (x+y+z)^{x+y+z}}{(x+y)^{x+y}(y+z)^{y+z}(z+x)^{z+x}}}\cdot a^{y+z}b^{z+x}c^{x+y} \tag{4.1}$$

with equality holding if and only if

$$\frac{a^2}{x(y+z)} = \frac{b^2}{y(z+x)} = \frac{c^2}{z(x+y)}.$$

The inequality (4.1) was obtained by Ke-Chang Yang in 1991 (see [5]).

Proof. This follows from Proposition 3.4, Law of Sines and $4RS = abc$. This completes the proof. ∎

Let $x = y = z = 1$, the inequality (4.1) is the well-known Pŏlyá-Szegó's inequality:

Proposition 4.2. *The following inequality holds*

$$S \le \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}, \tag{4.2}$$

with equality holding if and only if the triangle ABC is an equilateral triangle.

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