

**SOME CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE
INEQUALITIES FOR SEQUENCES OF OPERATORS IN
HILBERT SPACES**

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ABSTRACT. Some inequalities of Cauchy-Bunyakovsky-Schwarz type for sequences of bounded linear operators in Hilbert spaces and some applications are given.

1. INTRODUCTION

Let $(H; (\cdot, \cdot))$ be a real or complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators that map H into H .

We recall that a self-adjoint operator $A \in B(H)$ is positive in $B(H)$ iff $(Ax, x) \geq 0$ for any $x \in H$. The binary relation $A \geq B$ iff $A - B$ is a positive self-adjoint operator, is an *order relation* on $B(H)$. We remark that for any $A \in B(H)$ the operators $U := AA^*$ and $V := A^*A$ are positive self adjoint operators on H and $\|U\| = \|V\| = \|A\|^2$.

In [1], the author has proved the following inequality of Cauchy-Bunyakovsky-Schwarz type in the order of $B(H)$.

Theorem 1. *Let $A_1, \dots, A_n \in B(H)$ and $z_1, \dots, z_n \in \mathbb{K}$ (\mathbb{R}, \mathbb{C}). Then the following inequality holds:*

$$(1.1) \quad \sum_{i=1}^n |z_i|^2 \sum_{i=1}^n A_i A_i^* \geq \left(\sum_{i=1}^n z_i A_i \right) \left(\sum_{i=1}^n \bar{z}_i A_i^* \right) \geq 0.$$

Proof. For the sake of completeness, we give here a simple proof of this inequality.

For any $i, j \in \{1, \dots, n\}$ one has in the order of $B(H)$:

$$(\bar{z}_i A_j - \bar{z}_j A_i) (\bar{z}_i A_j - \bar{z}_j A_i)^* \geq 0,$$

that is,

$$(\bar{z}_i A_j - \bar{z}_j A_i) (z_i A_j^* - z_j A_i^*) \geq 0,$$

from where results

$$(1.2) \quad |z_i|^2 A_j A_j^* + |z_j|^2 A_i A_i^* \geq \bar{z}_i z_j A_j A_i^* + \bar{z}_j z_i A_i A_j^*$$

for any $i, j \in \{1, \dots, n\}$.

If we sum (1.2) over i from 1 to n we deduce

$$(1.3) \quad \left(\sum_{i=1}^n |z_i|^2 \right) A_j A_j^* + |z_j|^2 \left(\sum_{i=1}^n A_i A_i^* \right) \geq z_j A_j \left(\sum_{i=1}^n \bar{z}_i A_i^* \right) + \left(\sum_{i=1}^n z_i A_i \right) \bar{z}_j A_j^*,$$

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for any $j \in \{1, \dots, n\}$.

If we sum (1.3) over j from 1 to n , we deduce

$$(1.4) \quad \sum_{i=1}^n |z_i|^2 \sum_{j=1}^n A_j A_j^* + \sum_{j=1}^n |z_j|^2 \left(\sum_{i=1}^n A_i A_i^* \right) \\ \geq \sum_{j=1}^n z_j A_j \sum_{i=1}^n \bar{z}_i A_i^* + \left(\sum_{i=1}^n z_i A_i \right) \left(\sum_{j=1}^n \bar{z}_j A_j^* \right),$$

that is,

$$(1.5) \quad \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n A_k A_k^* \geq \sum_{k=1}^n z_k A_k \sum_{k=1}^n \bar{z}_k A_k^* = \left(\sum_{k=1}^n z_k A_k \right) \left(\sum_{k=1}^n z_k A_k \right)^* \geq 0,$$

and the theorem is proved. ■

The following version of the Cauchy-Bunyakovsky-Schwarz inequality for norms also holds [1].

Corollary 1. *With the assumptions in Theorem 1, one has*

$$(1.6) \quad \sum_{k=1}^n |z_k|^2 \left\| \sum_{k=1}^n A_k A_k^* \right\| \geq \left\| \sum_{k=1}^n z_k A_k \right\|^2.$$

Proof. The operators:

$$A := \sum_{k=1}^n |z_k|^2 \sum_{k=1}^n A_k A_k^*, \quad B := \left(\sum_{k=1}^n z_k A_k \right) \left(\sum_{k=1}^n \bar{z}_k A_k \right)^*$$

are obviously self-adjoint, positive and by (1.1), $A \geq B \geq 0$. Thus $\|A\| \geq \|B\|$ and since,

$$\|A\| = \sum_{k=1}^n |z_k|^2 \left\| \sum_{k=1}^n A_k A_k^* \right\|$$

and

$$\|B\| = \left\| \sum_{k=1}^n z_k A_k \right\|^2$$

the corollary is proved. ■

For other related results, see [2].

The main aim of this paper is to point out other inequalities similar to (1.6).

2. NORM INEQUALITIES

The following result holds.

Theorem 2. Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$. Then one has the inequalities:

$$(2.1) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \begin{cases} \max_{i=1, n} |\alpha_i|^2 \sum_{i=1}^n \|A_i\|^2 \\ \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|A_i\|^{2q} \right)^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{i=1, n} \|A_i\|^2 \\ \max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \\ + \begin{cases} \left[\left(\sum_{i=1}^n |\alpha_i|^r \right)^2 - \left(\sum_{i=1}^n |\alpha_i|^{2r} \right)^{\frac{1}{2}} \right] \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^s \right)^{\frac{1}{s}} \\ \text{if } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \left(\sum_{i=1}^n |\alpha_i|^2 \right) \right] \max_{1 \leq i \neq j \leq n} \|A_i A_j^*\|, \end{cases} \end{cases}$$

where (2.1) should be seen as all the 9 possible configurations.

Proof. We have

$$(2.2) \quad 0 \leq \left(\sum_{i=1}^n \alpha_i A_i \right) \left(\sum_{i=1}^n \alpha_i A_i \right)^* = \left(\sum_{i=1}^n \alpha_i A_i \right) \left(\sum_{j=1}^n \bar{\alpha}_j A_j^* \right) \\ = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j A_i A_j^* = \sum_{i=1}^n |\alpha_i|^2 A_i A_i^* + \sum_{1 \leq i \neq j \leq n} \alpha_i \bar{\alpha}_j A_i A_j^*.$$

Taking the norm in (2.2) and observing that $\|UU^*\| = \|U\|^2$ for any $U \in B(H)$, one has the inequality

$$(2.3) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 = \left\| \sum_{i=1}^n |\alpha_i|^2 A_i A_i^* + \sum_{1 \leq i \neq j \leq n} \alpha_i \bar{\alpha}_j A_i A_j^* \right\| \\ \leq \sum_{i=1}^n |\alpha_i|^2 \|A_i A_i^*\| + \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| \|A_i A_j^*\| \\ = \sum_{i=1}^n |\alpha_i|^2 \|A_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| \|A_i A_j^*\|.$$

Using Hölder's inequality, we may write that:

$$(2.4) \quad \sum_{i=1}^n |\alpha_i|^2 \|A_i\|^2 \leq \begin{cases} \max_{i=1,n} |\alpha_i|^2 \sum_{1 \leq i \neq j \leq n} \|A_i\|^2 \\ \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|A_i\|^{2q} \right)^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{i=1,n} \|A_i\|^2. \end{cases}$$

Also, Hölder's inequality for double sums produces

$$(2.5) \quad \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| \|A_i A_j^*\| \leq \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \\ \left(\sum_{1 \leq i \neq j \leq n} |\alpha_i|^r |\alpha_j|^r \right)^{\frac{1}{r}} \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^s \right)^{\frac{1}{s}} \\ \text{if } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \sum_{1 \leq i \neq j \leq n} |\alpha_i| |\alpha_j| \max_{1 \leq i \neq j \leq n} \|A_i A_j^*\|, \end{cases}$$

$$= \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \\ \left[\left(\sum_{i=1}^n |\alpha_i|^r \right)^2 - \left(\sum_{i=1}^n |\alpha_i|^{2r} \right)^{\frac{1}{2}} \right] \left(\sum_{i=1}^n \|A_i A_j^*\|^s \right)^{\frac{1}{s}} \\ \text{if } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \left(\sum_{i=1}^n |\alpha_i|^2 \right) \right] \max_{1 \leq i \neq j \leq n} \|A_i A_j^*\|, \end{cases}$$

Using (2.3) and (2.4), (2.5) one deduces the desired inequality (2.1). ■

The following corollaries are natural consequences.

Corollary 2. *With the assumptions of Theorem 2, one has the inequality*

$$(2.6) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\| \leq \max_{i=1,n} |\alpha_i| \left(\sum_{i,j=1}^n \|A_i A_j^*\| \right)^{\frac{1}{2}}.$$

Proof. Follows by the first line in (2.1) on taking into account that

$$\max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \leq \max_{i=1,n} |\alpha_i|^2,$$

and

$$\sum_{i,j=1}^n \|A_i A_j^*\| = \sum_{i=1}^n \|A_i\|^2 + \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|.$$

■

Corollary 3. *With the assumptions in Theorem 2, one has the inequality:*

$$(2.7) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\left(\sum_{i=1}^n \|A_i\|^{2q} \right)^{\frac{1}{q}} + (n-1) \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \right],$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality for positive numbers

$$\left(\sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2$$

we may write that

$$\begin{aligned} \left(\sum_{i=1}^n |\alpha_i|^p \right)^2 - \sum_{i=1}^n |\alpha_i|^{2p} &\leq n \sum_{i=1}^n |\alpha_i|^{2p} - \sum_{i=1}^n |\alpha_i|^{2p} \\ &= (n-1) \sum_{i=1}^n |\alpha_i|^{2p}. \end{aligned}$$

Now, using the second line in (2.1) for $r = p$, $s = q$, we deduce the desired result (2.7). ■

Corollary 4. *With the assumptions in Theorem 2, one has the inequality*

$$(2.8) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, \dots, n} \|A_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \right].$$

Proof. Follows by the third line of (2.1) on taking into account that

$$\left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right]^{\frac{1}{2}} \leq (n-1) \sum_{i=1}^n |\alpha_i|^2.$$

■

Another interesting particular case is embodied in the following corollary as well.

Corollary 5. *With the assumptions in Theorem 2, one has the inequality*

$$(2.9) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, \dots, n} \|A_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^2 \right)^{\frac{1}{2}} \right].$$

Proof. It is obvious that

$$\left[\left(\sum_{i=1}^n |\alpha_i|^2 \right)^2 - \sum_{i=1}^n |\alpha_i|^4 \right]^{\frac{1}{2}} \leq \sum_{i=1}^n |\alpha_i|^2.$$

Thus, combining the third line in the first bracket in (2.1) with the second line for $r = s = 2$ in the second bracket, the inequality (2.9) is obtained. ■

Remark 1. *If one is interested in obtaining bounds in terms of $\sum_{i=1}^n |\alpha_i|^2$, there are other possibilities as shown below. Obviously, since*

$$\max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \leq \max_{i=1, n} |\alpha_i|^2 \leq \sum_{i=1}^n |\alpha_i|^2.$$

then, by (2.1), in choosing the third line in the first bracket with the first line in the second bracket, one would obtain

$$(2.10) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, n} \|A_i\|^2 + \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \right].$$

Also, it is evident that

$$\left[\left(\sum_{i=1}^n |\alpha_i|^r \right)^2 - \left(\sum_{i=1}^n |\alpha_i|^{2r} \right)^{\frac{1}{2}} \right]^{\frac{1}{r}} \leq \left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{2}{r}}.$$

By the monotonicity of the power mean $\left(\frac{1}{n} \sum_{i=1}^n a_i^m \right)^{\frac{1}{m}}$ as a function of $m \in \mathbb{R}$, we have

$$\left(\frac{\sum_{i=1}^n |\alpha_i|^r}{n} \right)^{\frac{1}{r}} \leq \left(\frac{\sum_{i=1}^n |\alpha_i|^2}{n} \right)^{\frac{1}{2}}, \quad 1 < r \leq 2,$$

giving

$$\left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{2}{r}} \leq n^{\frac{2}{r}-1} \sum_{i=1}^n |\alpha_i|^2.$$

Thus, using the third line in the first bracket of (2.1) combined with the second line in the second bracket for $1 \leq r \leq 2$, $\frac{1}{s} + \frac{1}{r} = 1$, we deduce

$$(2.11) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, n} \|A_i\|^2 + n^{\frac{2}{r}-1} \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^s \right)^{\frac{1}{s}} \right].$$

Note that for $r = s = 2$, we recapture (2.9).

The following particular result also holds.

Proposition 1. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$ with the property that $A_i A_j^* = 0$ for any $i \neq j$, $i, j \in \{1, \dots, n\}$. Then one has the inequality;*

$$(2.12) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\| \leq \begin{cases} \max_{i=1, n} |\alpha_i| \left(\sum_{i=1}^n \|A_i\|^2 \right)^{\frac{1}{2}}, \\ \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{2p}} \left(\sum_{i=1}^n \|A_i\|^{2q} \right)^{\frac{1}{2q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \max_{i=1, n} \|A_i\|. \end{cases}$$

If by $M(\alpha, \mathbf{A})$ we denote any of the bounds provided by (2.1), (2.6), (2.7), (2.8), (2.9), (2.10) or (2.11), then we may state the following proposition as well.

Proposition 2. *Under the assumptions of Theorem 2, we have:*

(i) For any $x \in H$

$$(2.13) \quad \left\| \sum_{i=1}^n \alpha_i A_i x \right\|^2 \leq \|x\|^2 M(\boldsymbol{\alpha}, \mathbf{A}).$$

(ii) For any $x, y \in H$,

$$(2.14) \quad \left| \sum_{i=1}^n \alpha_i \langle A_i x, y \rangle \right|^2 \leq \|x\|^2 \|y\|^2 M(\boldsymbol{\alpha}, \mathbf{A}).$$

Proof. (i) Obviously,

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i A_i x \right\|^2 &= \left\| \left(\sum_{i=1}^n \alpha_i A_i \right) (x) \right\|^2 \leq \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \|x\|^2 \\ &\leq M(\boldsymbol{\alpha}, \mathbf{A}) \|x\|^2. \end{aligned}$$

(ii) We have

$$\left| \sum_{i=1}^n \alpha_i \langle A_i x, y \rangle \right|^2 = \left| \left\langle \sum_{i=1}^n \alpha_i A_i x, y \right\rangle \right|^2 = \left\| \sum_{i=1}^n \alpha_i A_i x \right\|^2 \|y\|^2,$$

which, by (i), gives the desired result (2.14). ■

3. INEQUALITIES FOR VECTORS IN HILBERT SPACES

We consider the non zero vectors $y_1, \dots, y_n \in H$. Define the operators

$$A_i : H \rightarrow H, \quad A_i x = \frac{(x, y_i)}{\|y_i\|} \cdot y_i, \quad i \in \{1, \dots, n\}.$$

Since

$$(3.1) \quad \|A_i\| = \sup_{\|x\|=1} \|A_i x\| = \sup_{\|x\|=1} |(x, y_i)| = \|y_i\|, \quad i \in \{1, \dots, n\}$$

then A_i are bounded linear operators in H . Also, since

$$(3.2) \quad (A_i x, x) = \left(\frac{(x, y_i) y_i}{\|y_i\|}, x \right) = \frac{|(x, y_i)|^2}{\|y_i\|} \geq 0, \quad x \in H, \quad i \in \{1, \dots, n\}$$

and

$$\begin{aligned} (A_i x, z) &= \left(\frac{(x, y_i) y_i}{\|y_i\|}, z \right) = \frac{(x, y_i) (y_i, z)}{\|y_i\|}, \\ (x, A_i z) &= \left(x, \frac{(z, y_i) y_i}{\|y_i\|} \right) = \frac{(x, y_i) \overline{(z, y_i)}}{\|y_i\|} = \frac{(x, y_i) (y_i, z)}{\|y_i\|}, \end{aligned}$$

giving

$$(3.3) \quad (A_i x, z) = (x, A_i z), \quad x, z \in H, \quad i \in \{1, \dots, n\},$$

we may conclude that A_i ($i = 1, \dots, n$) are positive self-adjoint operators on H .

Since, for any $x \in H$, one has

$$\begin{aligned} \|(A_i A_j)(x)\| &= \|(A_i)(A_j x)\| = \left\| A_i \left(\frac{(x, y_j) y_j}{\|y_j\|} \right) \right\| \\ &= \frac{|(x, y_j)|}{\|y_j\|} \|A_i y_j\| = \frac{|(x, y_j)|}{\|y_j\|} \cdot \frac{|(y_j, y_i)| \|y_j\|}{\|y_i\|} \\ &= \frac{|(x, y_j)| |(y_j, y_i)|}{\|y_i\|}, \quad i, j \in \{1, \dots, n\}, \end{aligned}$$

we deduce that

$$(3.4) \quad \|A_i A_j\| = \sup_{\|\alpha\|=1} \frac{|(x, y_j)| |(y_j, y_i)|}{\|y_i\|} = |(y_i, y_j)|; \quad i, j \in \{1, \dots, n\}.$$

If $(y_i)_{i=1, \dots, n}$ is an orthogonal family on H , then $\|A_i\| = 1$ and $A_i A_j = 0$ for $i, j \in \{1, \dots, n\}$, $i \neq j$.

The following inequality for vectors holds.

Theorem 3. *Let $x, y_1, \dots, y_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$(3.5) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \times \begin{cases} \max_{i=1, \dots, n} |\alpha_i|^2 \sum_{i=1}^n \|y_i\|^2 \\ \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|y_i\|^{2q} \right)^{\frac{1}{q}} & \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{i=1, \dots, n} \|y_i\|^2 \end{cases} + \|x\|^2 \times \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \\ \left[\left(\sum_{i=1}^n |\alpha_i|^r \right)^2 - \sum_{i=1}^n |\alpha_i|^{2r} \right]^r \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^s \right)^{\frac{1}{s}} & \text{if } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \left[\left(\sum_{i=1}^n |\alpha_i| \right)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(y_i, y_j)|. \end{cases}$$

Proof. Follows by Theorem 2 and Proposition 2, (i) on choosing $A_i = \frac{(\cdot, y_i)}{\|y_i\|} y_i$ and taking into account that $\|A_i\| = \|y_i\|$,

$$\|A_i A_j^*\| = |(y_i, y_j)|, \quad i, j \in \{1, \dots, n\}.$$

We omit the details. ■

Using Corollaries 2–5 and Remark 1, we may state the following particular inequalities:

$$(3.6) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\| \leq \|x\| \max_{i=1, n} |\alpha_i| \left(\sum_{i,j=1}^n |(y_i, y_j)| \right)^{\frac{1}{2}};$$

$$(3.7) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \left[\left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\left(\sum_{i=1}^n \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1) \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right] \right],$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$(3.8) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right];$$

$$(3.9) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, n} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right];$$

$$(3.10) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, n} \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right];$$

$$(3.11) \quad \left\| \sum_{i=1}^n \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=1, n} \|y_i\|^2 + n^{\frac{2}{r}-1} \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^s \right)^{\frac{1}{s}} \right],$$

where $1 < r \leq 2$, $\frac{1}{s} + \frac{1}{r} = 1$.

Remark 2. The choice $\alpha_i = \|y_i\|$ ($i = 1, \dots, n$) will produce some interesting bounds for

$$\left\| \sum_{i=1}^n (x, y_i) y_i \right\|^2.$$

We omit the details.

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