A GENERALISATION OF KUREPA'S INEQUALITY

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Abstract. A generalisation of Kurepa's inequality in inner product spaces that extends in its turn the de Bruijn refinement of the Cauchy-Buniakovsky-Schwarz inequality for sequences of real and complex numbers is given.

1. Introduction

In 1960, N.G. de Bruijn proved the following refinement of the celebrated Cauchy-Bunyakovsky-Schwarz (CBS) inequality, see [1] or [3, p. 48]:

Theorem 1. *Let $(a_1, \ldots, a_n)$ be an $n-$tuple of real numbers and $(z_1, \ldots, z_n)$ a $n-$tuple of complex numbers. Then*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right] \left(\leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} |z_k|^2\right). \tag{1.1}$$

Equality holds in (1.1) if and only if, for $k \in \{1, \ldots, n\}$, $a_k = \operatorname{Re}(\lambda z_k)$, where λ is a complex number such that $\lambda^2 \sum_{k=1}^{n} z_k^2$ is a nonnegative real number.

For various results in connection with the discrete $(CBS)-$inequality see the recent book [3].

In 1966, in an effort to extend this result to inner products, S. Kurepa considered the following setting (see [2]):

Let H be a *real* inner product space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the norm generated by the scalar product $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. The *complexification* $H_{\mathbb{C}}$ of H is defined as a complex linear space $H \times H$ of all ordered pairs (x, y) $(x, y \in H)$ endowed with the operations:

$$(x, y) + (x', y') := (x + x', y + y'), \tag{1.2}$$

$$(\sigma + i\tau) \cdot (x, y) := (\sigma x - \tau y, \tau x + \sigma y), \tag{1.3}$$

where $x, y, x', y' \in H$ and $\sigma, \tau \in \mathbb{R}$.

If $z = (x, y) \in H_{\mathbb{C}}$ and $z' \in (x', y') \in H_{\mathbb{C}}$ then we can define on $H_{\mathbb{C}}$ the following *complex scalar product*

$$\langle z, z' \rangle_{\mathbb{C}} := \langle x, x' \rangle + \langle y, y' \rangle + i \langle y, x' \rangle - i \langle x, y' \rangle \tag{1.4}$$

generating the norm

$$\|z\|_{\mathbb{C}} = \left(\|x\|^2 + \|y\|^2\right)^{1/2}, z = (x, y) \in H_{\mathbb{C}}.$$

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With this inner product $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle_{\mathbb{C}})$ becomes a complex inner product space. In this space we can also define the conjugate vector $\bar{z}$ of $z=(x,y)$ by

$$\bar{z}:=(x,-y)\,. \tag{1.5}$$

Similarly with the scalar case we denote

$$\operatorname{Re} z:=(x,0)\,,\qquad \operatorname{Im} z:=(0,y)\,. \tag{1.6}$$

Formally, we can write $z=x+iy=\operatorname{Re} z+i\operatorname{Im} z$ and $\bar{z}=x-iy=\operatorname{Re} z-i\operatorname{Im} z$.

We can now state the result due to Kurepa that provides a generalisation of the de Bruijn inequality.

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product and $(H_{\mathbb{C}};\langle\cdot,\cdot\rangle)$ its complexification. For any $a\in H$ and $z\in H_{\mathbb{C}}$ we have the inequality*

$$\left|\langle z,a\rangle_{\mathbb{C}}\right|^2\leq\|a\|^2\left[\frac{1}{2}\|z\|_{\mathbb{C}}^2+\frac{1}{2}\left|\langle z,\bar{z}\rangle_{\mathbb{C}}\right|\right]\left(\leq\|a\|^2\|z\|_{\mathbb{C}}^2\right). \tag{1.7}$$

If we take some particular examples of inner products, then we may state some interesting refinements of the $(CBS)-$inequality.

Corollary 1. *Let (S,Σ,μ) be a positive measure space and $f,g\in L_2(S,\Sigma,\mu)$. If f is a real function and g a complex function on S, then*

$$\begin{aligned}
&\left|\int_S f(s)\,g(s)\,d\mu(s)\right|^2 \\
&\leq\int_S f^2(s)\,d\mu(s)\left[\frac{1}{2}\int_S|g(s)|^2\,d\mu(s)+\frac{1}{2}\left|\int_S g^2(s)\,d\mu(s)\right|\right] \\
&\left(\leq\int_S f^2(s)\,d\mu(s)\int_S|g(s)|^2\,d\mu(s)\right).
\end{aligned} \tag{1.8}$$

Corollary 2. *If $(a_1,\dots,a_n)$ is an $n-$tuple of real numbers, $(z_1,\dots,z_n)$ an $n-$tuple of complex numbers and (A_{ij}) a positive definite $n\times n$ real matrix, then:*

$$\begin{aligned}
\left|\sum_{i,j=1}^n A_{ij}a_iz_j\right|^2&\leq\left(\sum_{i,j=1}^n A_{ij}a_ia_j\right)\left[\frac{1}{2}\sum_{i,j=1}^n A_{ij}z_i\overline{z_j}+\frac{1}{2}\left|\sum_{i,j=1}^n A_{ij}z_i\overline{z_j}\right|\right] \\
&\qquad\left(\leq\sum_{i,j=1}^n A_{ij}a_ia_j\sum_{i,j=1}^n A_{ij}z_i\overline{z_j}\right).
\end{aligned} \tag{1.9}$$

In this paper, a generalisation of Kurepa's inequality for the case of three vectors in real or complex inner product spaces is given.

Applications for discrete and integral inequalities extending the de Bruijn result (1.7) are also pointed out.

2. A Generalisation of Kurepa's Inequality

The following lemma is of interest.

Lemma 1. *Let $f:[0,2\pi]\to\mathbb{R}$ given by*

$$f(\alpha)=\lambda\sin^2\alpha+2\beta\sin\alpha\cos\alpha+\gamma\cos^2\alpha, \tag{2.1}$$

where $\lambda, \beta, \gamma \in \mathbb{R}$. *Then*

$$\sup_{\alpha \in [0,2\pi]} f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}\left[(\gamma - \lambda)^2 + 4\beta^2\right]^{\frac{1}{2}}. \tag{2.2}$$

Proof. Since

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \qquad \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}, \qquad 2\sin\alpha\cos\alpha = \sin 2\alpha,$$

hence f may be written as

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}(\gamma - \lambda)\cos 2\alpha + \beta \sin 2\alpha. \tag{2.3}$$

If $\beta = 0$, then (2.3) becomes

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}(\gamma - \lambda)\cos 2\alpha.$$

Obviously, in this case

$$\sup_{\alpha \in [0,2\pi]} f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{1}{2}|\gamma - \lambda| = \max(\gamma, \lambda).$$

If $\beta \neq 0$, then (2.3) becomes

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \beta\left[\sin 2\alpha + \frac{(\gamma - \lambda)}{\beta}\cos 2\alpha\right].$$

Let $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan\varphi = \frac{\gamma - \lambda}{2\beta}$. Then f can be written as

$$f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{\beta}{\cos\varphi}\sin(2\alpha + \varphi).$$

For this function, obviously

$$\sup_{\alpha \in [0,2\pi]} f(\alpha) = \frac{1}{2}(\lambda + \gamma) + \frac{|\beta|}{|\cos\varphi|}. \tag{2.4}$$

Since

$$\frac{\sin^2\varphi}{\cos^2\varphi} = \frac{(\gamma - \lambda)^2}{4\beta^2},$$

hence

$$\frac{1}{|\cos\varphi|} = \frac{\left[(\gamma - \lambda)^2 + 4\beta^2\right]^{\frac{1}{2}}}{2|\beta|},$$

and from (2.4) we deduce the desired result (2.2). ∎

We recall now, a functional $(\cdot,\cdot) : X \times X \to \mathbb{K}$, $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ is a *nonnegative Hermitian form* if

(h) $(x, x) \geq 0$ for all $x \in X$;
(hh) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ for all $\alpha, \beta \in \mathbb{K}$ and $x, y, z \in X$;
(hhh) $\overline{(y, x)} = (x, y)$ for all $x, y \in X$.

Such a functional $(\cdot,\cdot)$ generates the *semi-norm* $\|\cdot\|$ by

$$\|x\| := (x, x)^{\frac{1}{2}}, \qquad x \in X.$$

The following result holds.

Theorem 3. *Let X be a complex space and $(\cdot,\cdot)$ a nonnegative Hermitian form defined on X. If $x,y,z\in X$ are such that*

$$\operatorname{Im}(x,z)=\operatorname{Im}(y,z)=0, \tag{2.5}$$

then we have the inequality:

$$\begin{aligned}
&\operatorname{Re}^2(x,z)+\operatorname{Re}^2(y,z) \\
&=|(x+iy,z)|^2 \\
&\le \frac{1}{2}\left\{\|x\|^2+\|y\|^2+\left[\left(\|x\|^2-\|y\|^2\right)^2+4\operatorname{Re}^2(x,y)\right]^{\frac{1}{2}}\right\}\|z\|^2 \\
&\le \left(\|x\|^2+\|y\|^2\right)\|z\|^2 .
\end{aligned} \tag{2.6}$$

Proof. Obviously, by (2.5), we have

$$(x+iy,z)=\operatorname{Re}(x,z)+i\operatorname{Re}(y,z)$$

and the first part of (2.6) holds true.

Now, let $\varphi\in[0,2\pi]$ be such that

$$(x+iy,z)=e^{i\varphi}|(x+iy,z)| .$$

Then

$$|(x+iy,z)|=e^{-i\varphi}(x+iy,z)=\left(e^{-i\varphi}(x+iy),z\right).$$

Utilising the above inequality, we can write:

$$\begin{aligned}
|(x+iy,z)| &= \operatorname{Re}\left(e^{-i\varphi}(x+iy),z\right) \\
&= \operatorname{Re}\left((\cos\varphi-i\sin\varphi)(x+iy),z\right) \\
&= \operatorname{Re}(\cos\varphi\cdot x+\sin\varphi\cdot y-i\sin\varphi\cdot x+i\cos\varphi\cdot y,z) \\
&= \operatorname{Re}(\cos\varphi\cdot x+\sin\varphi\cdot y,z)+\operatorname{Im}(\sin\varphi\cdot x-\cos\varphi\cdot y,z) \\
&= \operatorname{Re}(\cos\varphi\cdot x+\sin\varphi\cdot y,z)+\sin\varphi\operatorname{Im}(x,z)-\cos\varphi\operatorname{Im}(y,z) \\
&= \operatorname{Re}(\cos\varphi\cdot x+\sin\varphi\cdot y,z),
\end{aligned}$$

and for the last equality, we have used the assumption (2.5).

Taking the square and using the Schwarz inequality for the nonnegative Hermitian form $(\cdot,\cdot)$, we have

$$\begin{aligned}
|(x+iy,z)|^2 &= \left[\operatorname{Re}(\cos\varphi\cdot x+\sin\varphi\cdot y,z)\right]^2 \\
&\le \|\cos\varphi\cdot x+\sin\varphi\cdot y\|^2\|z\|^2 .
\end{aligned} \tag{2.7}$$

On making use of Lemma 1, we have

$$\begin{aligned}
&\sup_{\varphi\in[0,2\pi]}\|\cos\varphi\cdot x+\sin\varphi\cdot y\|^2 \\
&=\sup_{\varphi\in[0,2\pi]}\left[\|x\|^2\cos^2\varphi+2\sin\varphi\cos\varphi\operatorname{Re}\langle x,y\rangle+\|y\|^2\sin^2\varphi\right] \\
&=\frac{1}{2}\left\{\|x\|^2+\|y\|^2+\left[\left(\|x\|^2-\|y\|^2\right)^2+4\operatorname{Re}^2(x,y)\right]^{\frac{1}{2}}\right\}
\end{aligned}$$

and the first inequality in (2.6) is also proved.

Observe that

$$\left(\|x\|^2-\|y\|^2\right)^2+4\operatorname{Re}^2(x,y)=\left(\|x\|^2-\|y\|^2\right)^2+4\left[\|x\|^2\|y\|^2-\operatorname{Re}^2(x,y)\right]$$
$$\leq\left(\|x\|^2-\|y\|^2\right)^2,$$

and the last part of (2.6) is proved. ∎

Remark 1. *Observe that if X is a real space and $(\cdot,\cdot)$ a nonnegative Hermitian form on X, then for any $x,y,z\in X$ one has:*

$$\begin{aligned}(x,z)^2+(y,z)^2&\\ \leq\frac{1}{2}&\left\{\|x\|^2+\|y\|^2+\left[\left(\|x\|^2-\|y\|^2\right)^2+4\operatorname{Re}^2(x,y)\right]^{\frac{1}{2}}\right\}\|z\|^2\\ \leq&\left(\|x\|^2+\|y\|^2\right)\|z\|^2.\end{aligned}\tag{2.8}$$

Remark 2. *If X is a real space, $(\cdot,\cdot)$ a nonnegative Hermitian form on X, $X_{\mathbb{C}}$ its complexification and $(\cdot,\cdot)_{\mathbb{C}}$ the corresponding nonnegative Hermitian form on X, then for $x,y\in X$ and $w:=x+iy\in X_{\mathbb{C}}$ and for $e\in X$ we have*

$$\operatorname{Im}(x,e)_{\mathbb{C}}=\operatorname{Im}(y,e)_{\mathbb{C}}=0,$$
$$\|w\|_{\mathbb{C}}^2=\|x\|^2+\|y\|^2,$$
$$|(w,\bar{w})_{\mathbb{C}}|=\left(\|x\|^2-\|y\|^2\right)^2+4(x,y)^2,$$

where $\bar{w}:=x-iy\in X_{\mathbb{C}}$.

Applying Theorem 3 for the complex space $X_{\mathbb{C}}$ and complex nonnegative Hermitian form $(\cdot,\cdot)_{\mathbb{C}}$, we deduce

$$|(w,e)_{\mathbb{C}}|^2\leq\frac{1}{2}\|e\|^2\left[\|w\|_{\mathbb{C}}^2+|(w,\bar{w})_{\mathbb{C}}|\right]\leq\|e\|^2\|w\|_{\mathbb{C}}^2,\tag{2.9}$$

which is a Kurepa type inequality for the Hermitian form $(\cdot,\cdot)_{\mathbb{C}}$.

Corollary 3. *Let x,y,z be as in Theorem 3. In addition, if $\operatorname{Re}(x,y)=0$, then*

$$\left[\operatorname{Re}^2(x,z)+\operatorname{Re}^2(y,z)\right]^{\frac{1}{2}}\leq\|z\|\max\{\|x\|,\|y\|\}.$$

Remark 3. *If X is a real space and $(\cdot,\cdot)$ a real nonnegative Hermitian form on X, then for any $x,y,z\in X$ with $(x,y)=0$ we have*

$$\left[(x,z)^2+(y,z)^2\right]^{\frac{1}{2}}\leq\|z\|\max\{\|x\|,\|y\|\}.$$

3. A Related Result

Utilising Lemma 1, we may state and prove the following result as well.

Theorem 4. *Let X be a real or complex linear space and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. Then we have the inequalities*

$$\begin{aligned}(3.1)\quad &\frac{1}{2}\Big\{|(v,t)|^2+|(w,t)|^2 \\ &+\Big[\left(|(v,t)|^2-|(w,t)|^2\right)^2+4\left(\operatorname{Re}(v,t)\operatorname{Re}(w,t)+\operatorname{Im}(v,t)\operatorname{Im}(w,t)\right)^2\Big]\Big\} \\ &\le \frac{1}{2}\|t\|^2\left\{\|v\|^2+\|w\|^2+\left[\left(\|v\|^2-\|w\|^2\right)^2+4\operatorname{Re}^2(v,w)\right]^{\frac{1}{2}}\right\} \\ &\le \left(\|v\|^2+\|w\|^2\right)\|t\|^2\end{aligned}$$

for all $v,w,t\in X$.

Proof. Observe that, by Schwarz's inequality for the nonnegative Hermitian form $(\cdot,\cdot)$, we have

$$(3.2)\qquad |(\cos\varphi\cdot v+\sin\varphi\cdot w,z)|^2\le\|\cos\varphi\cdot v+\sin\varphi\cdot w\|^2\|z\|^2$$

for any $\varphi\in[0,2\pi]$.

Since

$$\begin{aligned}I(\varphi)&:=\|\cos\varphi\cdot v+\sin\varphi\cdot w\|^2\\&=\|x\|^2\cos^2\varphi+2\sin\varphi\cos\varphi\operatorname{Re}(v,w)+\|w\|^2\sin^2\varphi\end{aligned}$$

hence, as in Theorem 3,

$$\sup_{\varphi\in[0,2\pi]}I(\varphi)=\frac{1}{2}\left\{\|v\|^2+\|w\|^2+\left[\left(\|v\|^2-\|w\|^2\right)^2+4\operatorname{Re}^2(v,w)\right]^{\frac{1}{2}}\right\}.$$

Also, denoting

$$\begin{aligned}J(\varphi)&:=|(v,z)\cos\varphi+(w,z)\sin\varphi|^2\\&=|(v,z)|^2\cos^2\varphi+2\sin\varphi\cos\varphi\operatorname{Re}\left[(v,z)\overline{(w,z)}\right]+|(w,z)|^2\sin^2\varphi,\end{aligned}$$

then, on applying Lemma 1, we deduce that

$$\begin{aligned}&\sup_{\varphi\in[0,2\pi]}J(\varphi)\\&=\frac{1}{2}\left\{|(v,t)|^2+|(w,t)|^2+\left[\left(|(v,t)|^2-|(w,t)|^2\right)^2+4\operatorname{Re}^2\left[(v,t)\overline{(w,t)}\right]\right]^{\frac{1}{2}}\right\}\end{aligned}$$

and, since

$$\operatorname{Re}\left[(v,t)\overline{(w,t)}\right]=\operatorname{Re}(v,t)\operatorname{Re}(w,t)+\operatorname{Im}(v,t)\operatorname{Im}(w,t),$$

hence, on taking the supremum in the inequality (3.2), we deduce the desired inequality (3.1). ∎

Remark 4. *In the real case, (3.1) provides the same inequality we obtained in (2.8).*

In the complex case, if we assume that $v,w,t\in X$ are such that

$$\operatorname{Re}(v,t)\operatorname{Re}(w,t)=-\operatorname{Im}(v,t)\operatorname{Im}(w,t),$$

then (3.1) becomes:

$$\max\left\{|(v,t)|^2, |(w,t)|^2\right\} \leq \frac{1}{2}\|t\|^2\left\{\|v\|^2+\|w\|^2+\left[\left(\|v\|^2-\|w\|^2\right)^2+4\operatorname{Re}^2(v,w)\right]^{\frac{1}{2}}\right\}. \tag{3.3}$$

4. Applications to Discrete Inequalities

Assume that $(K;\langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field $\mathbb{K}$. Assume also that $p_k \geq 0$, $k \in \mathbb{N}$ with $\sum_{k=1}^{\infty} p_k = 1$ and define

$$\ell_{\mathbf{p}}^2(K) := \left\{\mathbf{x} := (x_k)_{k\in\mathbb{N}} \;\middle|\; x_k \in K,\ k\in\mathbb{N} \text{ and } \sum_{k=1}^{\infty} p_k \|x_k\|^2 < \infty\right\}.$$

It is well known that $\ell_{\mathbf{p}}^2(K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathbf{p}}$ defined by

$$\langle x,y\rangle_{\mathbf{p}} := \sum_{k=1}^{\infty} p_k \langle x_k, y_k\rangle$$

and generating the norm

$$\|x\|_{\mathbf{p}} := \left(\sum_{k=1}^{\infty} p_k \|x_k\|^2\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

Utilising Theorem 3, one may state the following proposition.

Proposition 1. *Let* $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$, $\mathbf{y} = (y_k)_{k\in\mathbb{N}}$ *and* $\mathbf{z} = (z_k)_{k\in\mathbb{N}} \in \ell_{\mathbf{p}}^2(K)$ *such that*

$$\sum_{k=1}^{\infty} p_k \operatorname{Im}\langle x_k, z_k\rangle = \sum_{k=1}^{\infty} p_k \operatorname{Re}\langle y_k, z_k\rangle = 0.$$

Then we have the inequalities:

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} p_k \operatorname{Re}\langle x_k, y_k\rangle\right)^2 + \left(\sum_{k=1}^{\infty} p_k \operatorname{Re}\langle y_k, z_k\rangle\right)^2 \\
&\leq \frac{1}{2}\left\{\sum_{k=1}^{\infty} p_k\left(\|x_k\|^2+\|y_k\|^2\right) + \left[\left(\sum_{k=1}^{\infty} p_k\left(\|x_k\|^2-\|y_k\|^2\right)\right)^2\right.\right. \\
&\qquad\left.\left. + 4\left[\sum_{k=1}^{\infty} p_k \operatorname{Re}\langle x_k, y_k\rangle\right]^2\right]^{\frac{1}{2}}\right\}\sum_{k=1}^{\infty} p_k\|z_k\|^2 \\
&\leq \sum_{k=1}^{\infty} p_k\left(\|x_k\|^2+\|y_k\|^2\right)\sum_{k=1}^{\infty} p_k\|z_k\|^2.
\end{aligned} \tag{4.1}$$

In particular, if $\sum_{k=1}^{\infty} p_k \operatorname{Re}\langle y_k, z_k\rangle = 0$, *then*

$$\left[\left(\sum_{k=1}^{\infty} p_k \operatorname{Re}\langle x_k, y_k\rangle\right)^2 + \left(\sum_{k=1}^{\infty} p_k \operatorname{Re}\langle y_k, z_k\rangle\right)^2\right]^{\frac{1}{2}} \leq \max\left\{\left(\sum_{k=1}^{\infty} p_k \|x_k\|^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^{\infty} p_k \|y_k\|^2\right)^{\frac{1}{2}}\right\} \cdot \left(\sum_{k=1}^{\infty} p_k \|z_k\|^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Remark 5. *For the case of complex numbers, if* $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}, (z_k)_{k\in\mathbb{N}} \in \ell_{\mathbf{p}}^2(\mathbb{C})$ *are such that*

$$\sum_{k=1}^{\infty} p_k \operatorname{Re} x_k \cdot \operatorname{Im} z_k = \sum_{k=1}^{\infty} p_k \operatorname{Re} z_k \cdot \operatorname{Im} x_k \tag{4.3}$$

and

$$\sum_{k=1}^{\infty} p_k \operatorname{Re} y_k \cdot \operatorname{Im} z_k = \sum_{k=1}^{\infty} p_k \operatorname{Re} z_k \cdot \operatorname{Im} y_k,$$

then we have the inequalities

$$\begin{aligned} &\left[\sum_{k=1}^{\infty} p_k (\operatorname{Re} x_k \cdot \operatorname{Re} z_k + \operatorname{Im} x_k \cdot \operatorname{Im} z_k)\right]^2 \\ &\qquad + \left[\sum_{k=1}^{\infty} p_k (\operatorname{Re} y_k \cdot \operatorname{Re} z_k + \operatorname{Im} y_k \cdot \operatorname{Im} z_k)\right]^2 \\ &\leq \frac{1}{2}\left\{\sum_{k=1}^{\infty} p_k \left(\operatorname{Re}^2 x_k + \operatorname{Im}^2 x_k + \operatorname{Re}^2 y_k + \operatorname{Im}^2 y_k\right)\right. \\ &\qquad + \left(\sum_{k=1}^{\infty} p_k \left(\operatorname{Re}^2 x_k + \operatorname{Im}^2 x_k - \operatorname{Re}^2 y_k - \operatorname{Im}^2 y_k\right)\right)^2 \\ &\qquad \left. +4\left[\sum_{k=1}^{\infty} p_k (\operatorname{Re} x_k \cdot \operatorname{Re} y_k + \operatorname{Im} x_k \cdot \operatorname{Im} y_k)\right]^2\right\} \\ &\qquad \times \sum_{k=1}^{\infty} p_k \left(\operatorname{Re}^2 z_k + \operatorname{Im}^2 z_k\right) \\ &\leq \sum_{k=1}^{\infty} p_k \left(\operatorname{Re}^2 x_k + \operatorname{Im}^2 x_k + \operatorname{Re}^2 y_k + \operatorname{Im}^2 y_k\right) \sum_{k=1}^{\infty} p_k \left(\operatorname{Re}^2 z_k + \operatorname{Im}^2 z_k\right). \end{aligned} \tag{4.4}$$

Observe that if above, in particular, we assume that $\operatorname{Re} x_k = a_k$, $\operatorname{Im} x_k = 0$, $\operatorname{Re} y_k = b_k$, $\operatorname{Im} y_k = 0$, $\operatorname{Re} z_k = c_k$, $\operatorname{Im} z_k = 0$ *where* $a_k, b_k, c_k \in \mathbb{R}$, $k = \{1, \ldots, n\}$, *then*

(4.3) is satisfied, and, from (4.4), we deduce

$$\left(\sum_{k=1}^{\infty} p_k a_k c_k\right)^2 + \left(\sum_{k=1}^{\infty} p_k b_k c_k\right)^2$$
$$\leq \frac{1}{2}\left\{\sum_{k=1}^{\infty} p_k\left(a_k^2 + b_k^2\right) + \left[\left(\sum_{k=1}^{\infty} p_k\left(a_k^2 - b_k^2\right)\right)^2 + 4\left(\sum_{k=1}^{\infty} p_k a_k b_k\right)^2\right]^{\frac{1}{2}}\right\}\sum_{k=1}^{\infty} p_k c_k^2$$
$$\leq \sum_{k=1}^{\infty} p_k\left(a_k^2 + b_k^2\right)\sum_{k=1}^{\infty} p_k c_k^2,$$

which is the de Bruijn inequality for the complex numbers $w_k = a_k + ib_k$ and real numbers c_k, $k \in \mathbb{N}$, i.e.,

$$\left|\sum_{k=1}^{\infty} p_k c_k w_k\right|^2 \leq \frac{1}{2}\sum_{k=1}^{\infty} p_k c_k^2\left[\sum_{k=1}^{\infty} p_k |w_k|^2 + \left|\sum_{k=1}^{\infty} p_k w_k^2\right|^2\right]$$
$$\leq \sum_{k=1}^{\infty} p_k c_k^2 \sum_{k=1}^{\infty} p_k |w_k|^2,$$

provided $(c_k)_{k\in\mathbb{N}}, (w_k)_{k\in\mathbb{N}} \in \ell_{\mathbf{p}}^2(\mathbb{C})$.

Remark 6. *We note that the condition (4.3) may hold in other cases, for instance, if:*

$$\frac{\operatorname{Re} x_k}{\operatorname{Im} x_k} = \frac{\operatorname{Re} z_k}{\operatorname{Im} z_k} = \frac{\operatorname{Re} y_k}{\operatorname{Im} y_k},$$

for each $k \in \mathbb{N}$, provided $\operatorname{Im} x_k$, $\operatorname{Im} z_k$ and $\operatorname{Im} y_k$ are not zero for $k \in \mathbb{N}$.

5. Applications to Integral Inequalities

Assume that $(K; \langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho : [a,b] \subset \mathbb{R} \to [0,\infty)$ is a Lebesgue integrable function with $\int_a^b \rho(t)\,dt = 1$, then we may consider the space $L_\rho^2([a,b]; K)$ of all functions $f : [a,b] \to K$, that are Bochner measurable and $\int_a^b \rho(t)\|f(t)\|^2 dt < \infty$. It is well known that $L_\rho^2([a,b]; K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f, g\rangle_\rho := \int_a^b \rho(t)\langle f(t), g(t)\rangle\,dt$$

and generating the norm

$$\|f\|_\rho := \left(\int_a^b \rho(t)\|f(t)\|^2\,dt\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

Applying Theorem 3 for $L_\rho^2([a,b]; K)$, we may state the following proposition.

Proposition 2. *Let $f, g, h \in L_\rho^2([a,b]; K)$ such that*

$$\int_a^b \rho(t)\operatorname{Im}\langle f(t), h(t)\rangle\,dt = \int_a^b \rho(t)\operatorname{Im}\langle g(t), h(t)\rangle\,dt = 0.$$

Then we have the inequalities:

$$\begin{aligned}
&\left(\int_a^b \rho(t)\operatorname{Re}\langle f(t), h(t)\rangle dt\right)^2 + \left(\int_a^b \rho(t)\operatorname{Re}\langle g(t), h(t)\rangle dt\right)^2 \\
&\leq \frac{1}{2}\left\{\int_a^b \rho(t)\left(\|f(t)\|^2 + \|g(t)\|^2\right) dt\right. \\
&+\left.\left[\left(\int_a^b \rho(t)\left(\|f(t)\|^2 - \|g(t)\|^2\right) dt\right)^2 + 4\left(\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt\right)^2\right]^{\frac{1}{2}}\right\} \\
&\times \int_a^b \rho(t)\|h(t)\|^2 dt \\
&\leq \int_a^b \rho(t)\left(\|f(t)\|^2 + \|g(t)\|^2\right) dt \cdot \int_a^b \rho(t)\|h(t)\|^2 dt.
\end{aligned}$$

In particular, if

$$\int_a^b \rho(t)\operatorname{Re}\langle f(t), g(t)\rangle dt = 0$$

then

$$\begin{aligned}
&\left[\left(\int_a^b \rho(t)\operatorname{Re}\langle f(t), h(t)\rangle dt\right)^2 + \left(\int_a^b \rho(t)\operatorname{Re}\langle g(t), h(t)\rangle dt\right)^2\right]^{\frac{1}{2}} \\
&\leq \max\left\{\left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}, \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}}\right\}\left(\int_a^b \rho(t)\|h(t)\|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

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