INEQUALITIES FOR ORTHORNORMAL FAMILIES OF VECTORS IN INNER PRODUCT SPACES RELATED TO BUZANO'S, RICHARD'S AND KUREPA'S RESULTS

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ABSTRACT. Some inequalities for families of orthornormal vectors in inner product spaces that are related with Buzano's, Richard's and Kurepa's results are given.

1. INTRODUCTION

In [3], M.L. Buzano obtained the following extension of the celebrated Schwarz's inequality in a real or complex inner product space $(H;\langle\cdot,\cdot\rangle)$:

$$|\langle a,x\rangle\langle x,b\rangle| \leq \frac{1}{2}\left[\|a\|\|b\| + |\langle a,b\rangle|\right]\|x\|^2, \tag{1.1}$$

for any $a,b,x\in H$.

It is clear that the above inequality becomes, for $a=b$, the Schwarz's inequality

$$|\langle a,x\rangle|^2 \leq \|a\|^2\|x\|^2, \quad a,x\in H; \tag{1.2}$$

in which the equality holds if and only if there exists a scalar $\lambda\in\mathbb{K}$ $(\mathbb{R},\mathbb{C})$ so that $x=\lambda a$.

As noted by M. Fujii and I. Kubo in [5], where they provided a simple proof of (1.1) on using orthogonal projection arguments, the case of equality holds in (1.1) if

$$x = \begin{cases} \alpha\left(\frac{a}{\|a\|} + \frac{\langle a,b\rangle}{|\langle a,b\rangle|}\cdot\frac{b}{\|b\|}\right), & \text{when} \quad \langle a,b\rangle \neq 0 \\ \alpha\left(\frac{a}{\|a\|} + \beta\cdot\frac{b}{\|b\|}\right), & \text{when} \quad \langle a,b\rangle = 0, \end{cases} \tag{1.3}$$

where $\alpha,\beta\in\mathbb{K}$.

As noted by T. Precupanu in [8], independently of Buzano, U. Richard [9] obtained the following similar inequality holding in real inner product spaces:

$$\frac{1}{2}\|a\|^2\left[\langle a,b\rangle - \|a\|\|b\|\right] \leq \langle a,x\rangle\langle x,b\rangle \leq \frac{1}{2}\|x\|^2\left[\langle a,b\rangle - \|a\|\|b\|\right]. \tag{1.4}$$

In [7], J. Pečarić gave a simple direct proof of (1.4) without mentioning the work of either Buzano of Richard, but tracked down the result, in a particular form, to an earlier paper of C. Blatter [1].

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In [4], the author has improved Buzano's inequality by showing that, in fact, one has:

$$|\langle a,x\rangle\langle x,b\rangle| \le \left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\langle a,b\rangle\|x\|^2\right| + \frac{1}{2}|\langle a,b\rangle|\|x\|^2 \le \frac{1}{2}\left[\|a\|\|b\| + |\langle a,b\rangle|\right]\|x\|^2. \tag{1.5}$$

In the same paper, the author has also improved the celebrated Kurepa inequality for the complexification of a real inner product space, namely, the inequality [6]

$$|\langle z,x\rangle_{\mathbb{C}}|^2 \le \frac{1}{2}\|x\|^2\left[\|z\|_{\mathbb{C}}^2 + |\langle z,\bar{z}\rangle_{\mathbb{C}}|\right] \le \|x\|^2\|z\|_{\mathbb{C}}^2, \tag{1.6}$$

where $x \in H$, $(H;\langle\cdot,\cdot\rangle)$ is a real space, $z \in H_{\mathbb{C}}$, $(H_{\mathbb{C}},\langle\cdot,\cdot\rangle_{\mathbb{C}})$ is the complexification of $(H;\langle\cdot,\cdot\rangle)$, and $\bar{z}$ is the conjugate vector of z.

The refinement of Kurepa's result (1.6) obtained in [4] is as follows:

$$\begin{aligned} |\langle z,x\rangle_{\mathbb{C}}|^2 &\le \left|\langle z,x\rangle_{\mathbb{C}}^2 - \frac{1}{2}\langle z,\bar{z}\rangle_{\mathbb{C}}\|x\|^2\right| + \frac{1}{2}|\langle z,\bar{z}\rangle_{\mathbb{C}}|\|x\|^2 \\ &\le \frac{1}{2}\|x\|^2\left[\|z\|_{\mathbb{C}}^2 + |\langle z,\bar{z}\rangle_{\mathbb{C}}|\right] \\ &\le \|x\|^2\|z\|_{\mathbb{C}}^2, \end{aligned} \tag{1.7}$$

with the same assumptions as above.

The main aim of the present paper is to obtain similar results for families of orthonormal vectors in $(H;\langle\cdot,\cdot\rangle)$, real or complex space, that are naturally connected with the celebrated Bessel inequality.

2. A Generalisation for Orthonormal Families

We say that the finite family $\{e_i\}_{i\in I}$ (I is finite) of vectors is *orthonormal* if $\langle e_i,e_j\rangle = 0$ if $i,j\in I$ with $i\ne j$ and $\|e_i\| = 1$ for each $i\in I$. The following result may be stated:

Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i\in I}$ a finite orthonormal family in H. Then for any $a,b\in H$, one has the inequality:*

$$\left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle - \frac{1}{2}\langle a,b\rangle\right| \le \frac{1}{2}\|a\|\|b\|. \tag{2.1}$$

The case of equality holds in (2.1) if and only if

$$\sum_{i\in I}\langle a,e_i\rangle e_i = \frac{1}{2}a + \left(\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle - \frac{1}{2}\langle a,b\rangle\right)\cdot\frac{b}{\|b\|^2}. \tag{2.2}$$

Proof. It is well known that, for $e\ne 0$ and $f\in H$, the following identity holds:

$$\frac{\|f\|^2\|e\|^2 - |\langle f,e\rangle|^2}{\|e\|^2} = \left\|f - \frac{\langle f,e\rangle e}{\|e\|^2}\right\|^2. \tag{2.3}$$

Therefore, in Schwarz's inequality

$$|\langle f,e\rangle|^2 \le \|f\|^2\|e\|^2, \quad f,e\in H; \tag{2.4}$$

the case of equality, for $e \neq 0$, holds if and only if

$$f = \frac{\langle f, e\rangle e}{\|e\|^2}.$$

Let $f := 2\sum_{i\in I}\langle a, e_i\rangle e_i - a$ and $e := b$. Then, by Schwarz's inequality (2.4), we may state that

$$\left|\left\langle 2\sum_{i\in I}\langle a, e_i\rangle e_i - a, b\right\rangle\right|^2 \leq \left\|2\sum_{i\in I}\langle a, e_i\rangle e_i - a\right\|^2 \|b\|^2 \tag{2.5}$$

with equality, for $b \neq 0$, if and only if

$$2\sum_{i\in I}\langle a, e_i\rangle e_i - a = \left\langle 2\sum_{i\in I}\langle a, e_i\rangle e_i - a, b\right\rangle \frac{b}{\|b\|^2}. \tag{2.6}$$

Since

$$\left\langle 2\sum_{i\in I}\langle a, e_i\rangle e_i - a, b\right\rangle = 2\sum_{i\in I}\langle a, e_i\rangle\langle e_i, b\rangle - \langle a, b\rangle$$

and

$$\begin{aligned}\left\|2\sum_{i\in I}\langle a, e_i\rangle e_i - a\right\|^2 &= 4\left\|\sum_{i\in I}\langle a, e_i\rangle e_i\right\|^2 - 4\operatorname{Re}\left\langle\sum_{i\in I}\langle a, e_i\rangle e_i, a\right\rangle + \|a\|^2 \\ &= 4\sum_{i\in I}|\langle a, e_i\rangle|^2 - 4\sum_{i\in I}|\langle a, e_i\rangle|^2 + \|a\|^2 \\ &= \|a\|^2,\end{aligned}$$

hence by (2.5) we deduce the desired inequality (2.1).

Finally, as (2.2) is equivalent to

$$\sum_{i\in I}\langle a, e_i\rangle e_i - \frac{a}{2} = \left(\sum_{i\in I}\langle a, e_i\rangle\langle e_i, b\rangle - \frac{1}{2}\langle a, b\rangle\right)\frac{b}{\|b\|^2},$$

hence the equality holds in (2.1) if and only if (2.2) is valid. ∎

The following result is well known in the literature as Bessel's inequality

$$\sum_{i\in I}|\langle x, e_i\rangle|^2 \leq \|x\|^2, \quad x \in H, \tag{2.7}$$

where, as above, $\{e_i\}_{i\in I}$ is a finite orthonormal family in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

If one chooses $a = b = x$ in (2.1), then one gets the inequality

$$\left|\sum_{i\in I}|\langle x, e_i\rangle|^2 - \frac{1}{2}\|x\|^2\right| \leq \frac{1}{2}\|x\|^2,$$

which is obviously equivalent to Bessel's inequality (2.7). Therefore, the inequality (2.1) may be regarded as a generalisation of Bessel's inequality as well.

Utilising the Bessel and Cauchy-Bunyakovsky-Schwarz inequalities, one may state that

$$\left|\sum_{i\in I}\langle a, e_i\rangle\langle e_i, b\rangle\right| \leq \left[\sum_{i\in I}|\langle a, e_i\rangle|^2\sum_{i\in I}|\langle b, e_i\rangle|^2\right]^{\frac{1}{2}} \leq \|a\|\|b\| \tag{2.8}$$

A different refinement of the inequality between the first and the last term in (2.8) is incorporated in the following:

Corollary 1. *With the assumption of Theorem 1, we have*

$$\left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle\right| \le \left|\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle-\frac{1}{2}\langle a,b\rangle\right|+\frac{1}{2}\left|\langle a,b\rangle\right| \le \frac{1}{2}\left[\|a\|\,\|b\|+\left|\langle a,b\rangle\right|\right] \le \|a\|\,\|b\|. \tag{2.9}$$

Remark 1. *If the space $(H;\langle\cdot,\cdot\rangle)$ is real, then, obviously, (2.1) is equivalent to:*

$$\frac{1}{2}\left(\langle a,b\rangle-\|a\|\,\|b\|\right)\le\sum_{i\in I}\langle a,e_i\rangle\langle e_i,b\rangle\le\frac{1}{2}\left[\|a\|\,\|b\|+\langle a,b\rangle\right]. \tag{2.10}$$

Remark 2. *It is obvious that if the family comprises of only a single element $e=\frac{x}{\|x\|}$, $x\in H$, $x\neq 0$, then from (2.9) we recapture the refinement of Buzano's inequality incorporated in (1.5) while from (2.10) we deduce Richard's result from (1.4).*

The following corollary of Theorem 1 is of interest as well:

Corollary 2. *Let $\{e_i\}_{i\in I}$ be a finite orthonormal family in $(H;\langle\cdot,\cdot\rangle)$. If $x,y\in H\setminus\{0\}$ are such that there exists the constants m_i,n_i, M_i, $N_i\in\mathbb{R}$, $i\in I$ such that:*

$$-1\le m_i\le\frac{\operatorname{Re}\langle x,e_i\rangle}{\|x\|}\cdot\frac{\operatorname{Re}\langle y,e_i\rangle}{\|y\|}\le M_i\le 1,\quad i\in I \tag{2.11}$$

and

$$-1\le n_i\le\frac{\operatorname{Im}\langle x,e_i\rangle}{\|x\|}\cdot\frac{\operatorname{Im}\langle y,e_i\rangle}{\|y\|}\le N_i\le 1,\quad i\in I \tag{2.12}$$

then

$$2\sum_{i\in I}(m_i+n_i)-1\le\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\,\|y\|}\le 1+2\sum_{i\in I}(M_i+N_i). \tag{2.13}$$

Proof. Using Theorem 1 and the fact that for any complex number z, $|z|\ge|\operatorname{Re}z|$, we have

$$\left|\sum_{i\in I}\operatorname{Re}\left[\langle x,e_i\rangle\langle e_i,y\rangle\right]-\frac{1}{2}\operatorname{Re}\langle x,y\rangle\right|\le\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle-\frac{1}{2}\langle x,y\rangle\right| \le\frac{1}{2}\|x\|\,\|y\|. \tag{2.14}$$

Since

$$\operatorname{Re}\left[\langle x,e_i\rangle\langle e_i,y\rangle\right]=\operatorname{Re}\langle x,e_i\rangle\operatorname{Re}\langle y,e_i\rangle+\operatorname{Im}\langle x,e_i\rangle\operatorname{Im}\langle y,e_i\rangle,$$

hence by (2.14) we have:

$$-\frac{1}{2}\|x\|\,\|y\|+\frac{1}{2}\operatorname{Re}\langle x,y\rangle \le\sum_{i\in I}\operatorname{Re}\langle x,e_i\rangle\operatorname{Re}\langle y,e_i\rangle+\sum_{i\in I}\operatorname{Im}\langle x,e_i\rangle\operatorname{Im}\langle y,e_i\rangle \le\frac{1}{2}\|x\|\,\|y\|+\frac{1}{2}\operatorname{Re}\langle x,y\rangle. \tag{2.15}$$

Utilising the assumptions (2.11) and (2.12), we have

$$\sum_{i\in I} m_i \le \sum_{i\in I} \frac{\operatorname{Re}\langle x, e_i\rangle \operatorname{Re}\langle y, e_i\rangle}{\|x\| \|y\|} \le \sum_{i\in I} M_i \tag{2.16}$$

and

$$\sum_{i\in I} n_i \le \sum_{i\in I} \frac{\operatorname{Im}\langle x, e_i\rangle \operatorname{Im}\langle y, e_i\rangle}{\|x\| \|y\|} \le \sum_{i\in I} N_i. \tag{2.17}$$

Finally, on making use of (2.15) – (2.17), we deduce the desired result (2.13). ∎

Remark 3. *By Schwarz's inequality, is it obvious that, in general,*

$$-1 \le \frac{\operatorname{Re}\langle x, y\rangle}{\|x\| \|y\|} \le 1.$$

Consequently, the left inequality in (2.13) is of interest if $\sum_{i\in I}(m_i + n_i) > 0$, *while the right inequality in (2.13) is of interest if* $\sum_{i\in I}(M_i + N_i) < 0$.

3. Refinements of Kurepa's Inequality

Let $(H; \langle\cdot,\cdot\rangle)$ be a real inner product space generating the norm $\|\cdot\|$. The *complexification* $H_{\mathbb{C}}$ of H is defined as a complex linear space $H \times H$ of all ordered pairs (x, y), $x, y \in H$ endowed with the operations:

$$(x, y) + (x', y') := (x + x', y + y'), \qquad x, x', y, y' \in H;$$
$$(\sigma + i\tau)\cdot(x, y) := (\sigma x - \tau y, \tau x + \sigma y), \qquad x, y \in H \text{ and } \sigma, \tau \in \mathbb{R}.$$

On $H_{\mathbb{C}} := H \times H$, endowed with the above operations, one can now canonically define the *scalar product* $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ by:

$$\langle z, z'\rangle_{\mathbb{C}} := \langle x, x'\rangle + \langle y, y'\rangle + i\left[\langle x', y\rangle - \langle x, y'\rangle\right] \tag{3.1}$$

where $z = (x, y)$, $z' = (x', y') \in H_{\mathbb{C}}$. Obviously,

$$\|z\|_{\mathbb{C}}^2 = \|x\|^2 + \|y\|^2, \qquad z = (x, y) \in H_{\mathbb{C}}.$$

One can also define the *conjugate* of a vector $z = (x, y)$ by $\bar{z} := (x, -y)$. It is easy to see that, the elements of $H_{\mathbb{C}}$, under defined operations, behave as formal "complex" combinations $x + iy$ with $x, y \in H$. Because of this, we may write $z = x + iy$ instead of $z = (x, y)$. Thus, $\bar{z} = x - iy$. Under this setting, S. Kurepa [6] proved the following refinement of Schwarz's inequality:

$$|\langle a, z\rangle_{\mathbb{C}}|^2 \le \frac{1}{2}\|a\|^2\left[\|z\|_{\mathbb{C}}^2 + |\langle z, \bar{z}\rangle_{\mathbb{C}}|\right] \le \|a\|^2\|z\|_{\mathbb{C}}^2, \tag{3.2}$$

for any $a \in H$ and $z \in H_{\mathbb{C}}$.

This was motivated by generalising the de Bruijn result for sequences of real and complex numbers obtained in [2].

The following result holds.

Theorem 2. *Let* $\{e_j\}_{j\in I}$ *be a finite orthonormal family in the real inner product space* $(H; \langle\cdot,\cdot\rangle)$. *Then for any* $w \in H_{\mathbb{C}}$, *where* $(H_{\mathbb{C}}; \langle\cdot,\cdot\rangle_{\mathbb{C}})$ *is the complexification of*

$(H; \langle \cdot, \cdot \rangle)$, *one has the following Bessel's type inequality:*

$$\left| \sum_{j \in I} \langle w, e_j \rangle^2 \right| \leq \left| \sum_{j \in I} \langle w, e_j \rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle w, \bar{w} \rangle_{\mathbb{C}} \right| + \frac{1}{2} \left| \langle w, \bar{w} \rangle_{\mathbb{C}} \right| \tag{3.3}$$
$$\leq \frac{1}{2} \left[\|w\|_{\mathbb{C}}^2 + \left| \langle w, \bar{w} \rangle_{\mathbb{C}} \right| \right] \leq \|w\|_{\mathbb{C}}^2 .$$

Proof. Define $f_j \in H_{\mathbb{C}}$, $f_j := (e_j, 0)$, $j \in I$. For any $k, j \in I$ we have

$$\langle f_i, f_j \rangle_{\mathbb{C}} = \langle (e_k, 0), (e_j, 0) \rangle_{\mathbb{C}} = \langle e_k, e_j \rangle = \delta_{kj},$$

therefore $\{f_j\}_{j \in I}$ is an orthonormal family in $(H_{\mathbb{C}}; \langle \cdot, \cdot \rangle_{\mathbb{C}})$.

If we apply Theorem 1 for $(H_{\mathbb{C}}; \langle \cdot, \cdot \rangle_{\mathbb{C}})$, $a = w$, $b = \bar{w}$, we may write:

$$\left| \sum_{j \in I} \langle w, e_j \rangle_{\mathbb{C}} \langle e_j, \bar{w} \rangle_{\mathbb{C}} - \frac{1}{2} \langle w, \bar{w} \rangle_{\mathbb{C}} \right| \leq \frac{1}{2} \|w\|_{\mathbb{C}} \|\bar{w}\|_{\mathbb{C}} . \tag{3.4}$$

However, for $w := (x, y) \in H_{\mathbb{C}}$, we have $\bar{w} = (x, -y)$ and

$$\langle e_j, \bar{w} \rangle_{\mathbb{C}} = \langle (e_j, 0), (x, -y) \rangle_{\mathbb{C}} = \langle e_j, x \rangle - i \langle e_j, -y \rangle = \langle e_j, x \rangle + i \langle e_j, y \rangle$$

and

$$\langle w, e_j \rangle_{\mathbb{C}} = \langle (x, y), (e_j, 0) \rangle_{\mathbb{C}} = \langle e_j, x \rangle - i \langle e_j, -y \rangle = \langle x, e_j \rangle + i \langle e_j, y \rangle$$

for any $j \in I$. Thus $\langle e_j, \bar{w} \rangle = \langle w, e_j \rangle$ for each $j \in I$ and since

$$\|w\|_{\mathbb{C}} = \|\bar{w}\|_{\mathbb{C}} = \left(\|x\|^2 + \|y\|^2 \right)^{\frac{1}{2}},$$

we get from (3.4) that

$$\left| \sum_{j \in I} \langle w, e_j \rangle_{\mathbb{C}}^2 - \frac{1}{2} \langle w, \bar{w} \rangle_{\mathbb{C}} \right| \leq \frac{1}{2} \|w\|_{\mathbb{C}}^2 . \tag{3.5}$$

Now, observe that the first inequality in (3.3) follows by the triangle inequality, the second is an obvious consequence of (3.5) and the last one is derived from Schwarz's result. ∎

Remark 4. *If the family* $\{e_j\}_{j \in I}$ *contains only a single element* $e = \frac{x}{\|x\|}$, $x \in H$, $x \neq 0$, *then from (3.3) we deduce (1.7), which, in its turn, provides a refinement of Kurepa's inequality (3.2).*

4. An Application for $L_2[-\pi, \pi]$

It is well known that in the Hilbert space $L_2[-\pi, \pi]$ of all functions $f : [-\pi, \pi] \to \mathbb{C}$ with the property that f is Lebesgue measurable on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$, the set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \dots, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt, \dots \right\}$$

is orthonormal.

If by $\operatorname{trig} t$, we denote either $\sin t$ or $\cos t$, $t \in [-\pi, \pi]$, then on using the results from Sections 2 and 3, we may state the following inequality:

$$\left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \operatorname{trig}(kt)\, dt \cdot \int_{-\pi}^{\pi} \overline{g(t)} \operatorname{trig}(kt)\, dt - \frac{1}{2} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt \right|^2 \leq \frac{1}{4} \int_{-\pi}^{\pi} |f(t)|^2 \, dt \int_{-\pi}^{\pi} |g(t)|^2 \, dt, \tag{4.1}$$

where all $\operatorname{trig}(kt)$ is either $\sin kt$ or $\cos kt$, $k \in \{1, \dots, n\}$ and $f \in L_2[-\pi, \pi]$.

This follows by Theorem 1.

If one uses Corollary 1, then one can state the following chain of inequalities

$$\begin{aligned} &\left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \operatorname{trig}(kt)\, dt \cdot \int_{-\pi}^{\pi} \overline{g(t)} \operatorname{trig}(kt)\, dt \right| \\ &\leq \left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \operatorname{trig}(kt)\, dt \cdot \int_{-\pi}^{\pi} \overline{g(t)} \operatorname{trig}(kt)\, dt - \frac{1}{2} \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt \right| \\ &\qquad + \frac{1}{2} \left| \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt \right| \\ &\leq \frac{1}{2} \left[\left(\int_{-\pi}^{\pi} |f(t)|^2 \, dt \int_{-\pi}^{\pi} |g(t)|^2 \, dt \right)^{\frac{1}{2}} + \left| \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt \right| \right] \\ &\leq \left(\int_{-\pi}^{\pi} |f(t)|^2 \, dt \int_{-\pi}^{\pi} |g(t)|^2 \, dt \right)^{\frac{1}{2}}, \end{aligned} \tag{4.2}$$

where $f \in L_2[-\pi, \pi]$.

Finally, by employing Theorem 2, we may state:

$$\begin{aligned} &\frac{1}{\pi} \left| \sum_{k=1}^{n} \left[\int_{-\pi}^{\pi} f(t) \operatorname{trig}(kt)\, dt \right]^2 \right| \\ &\leq \left| \frac{1}{\pi} \sum_{k=1}^{n} \left[\int_{-\pi}^{\pi} f(t) \operatorname{trig}(kt)\, dt \right]^2 - \frac{1}{2} \int_{-\pi}^{\pi} f^2(t)\, dt \right| + \frac{1}{2} \left| \int_{-\pi}^{\pi} f^2(t)\, dt \right| \\ &\leq \frac{1}{2} \left[\int_{-\pi}^{\pi} |f(t)|^2 \, dt + \left| \int_{-\pi}^{\pi} f^2(t)\, dt \right| \right] \leq \int_{-\pi}^{\pi} |f(t)|^2 \, dt, \end{aligned}$$

where $f \in L_2[-\pi, \pi]$.

References

[1] C. BLATTER, Zur Riemannschen Geometrie im Grossen auf dem Möbiusband. (German) *Compositio Math.* **15** (1961) 88–107.

[2] N.G. de BRUIJN, Problem 12, *Wisk. Opgaven*, **21** (1960), 12-14.

[3] M.L. BUZANO, Generalizzazione della disiguaglianza di Cauchy-Schwarz (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405-409 (1974).

[4] S.S. DRAGOMIR, Refinements of Buzano's and Kurepa's inequalities in inner product spaces, Preprint, *RGMIA Res. Rep. Coll.,* **7** (2004).

[5] M. FUJII and F. KUBO, Buzano's inequality and bounds for roots of algebraic equations, *Proc. Amer. Math. Soc.,* **117**(2) (1993), 359-361.

[6] S. KUREPA, On the Buniakowsky-Cauchy-Schwarz inequality, *Glasnick Mathematički,* **1**(**21**)(2) (1966), 147-158.

[7] J.E. PEČARIĆ, On some classical inequalities in unitary spaces, *Mat. Bilten*, (Macedonia) **16** (1992), 63-72.

[8] T. PRECUPANU, On a generalisation of Cauchy-Buniakowski-Schwarz inequality, *Anal. St. Univ. "Al. I. Cuza" Iaşi,* **22**(2) (1976), 173-175.

[9] U. RICHARD, Sur des inégalités du type Wirtinger et leurs application aux équationes différentielles ordinaires, Colloquium of Analysis held in Rio de Janeiro, August, 1972, pp. 233-244.

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