On an Extension of Variant of Jensen's Inequality for Convex Function

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Abstract

In this paper, we extend a variant of Jensen's inequality for convex function.

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1 Introduction and Main results

Let f be a given convex function defined on a non-empty interval $\mathbb{I} \subset \mathbb{R}$. And let $x_i \in \mathbb{I}$, $\omega_i > 0 (i = 1, 2, \cdots, n; n \geq 2)$, $\sum\limits_{i=1}^{n} \omega_i = 1$. It is well-known that the following Jensen's inequality [1,2] holds

$$f\left(\sum_{i=1}^{n} \omega_i x_i\right) \leq \sum_{i=1}^{n} \omega_i f(x_i). \tag{1.1}$$

Under the above hypotheses and $x_1 \leq x_2 \leq \cdots \leq x_n$, A. M. Mercer [3] gave a variant of inequality (1.1):

$$f\left(x_1 + x_n - \sum_{i=1}^{n} \omega_i x_i\right) \leq f(x_1) + f(x_n) - \sum_{i=1}^{n} \omega_i f(x_i). \tag{1.2}$$

In this note, our purpose is to prove the following extension of inequality (1.2):

Theorem 1.1. *Let f be a convex function on $\mathbb{I}$ containing x_i, and let $t_i \in \mathbb{R}/\{0\} (i = 1, 2, \cdots, n; n \geq 2)$. If $\{x_1, x_2 \cdots, x_n\}$ be monotonic sequence, and*

$$0 \leq \sum_{i=k}^{n} t_i \leq \sum_{i=1}^{n} t_i = T_n, \quad T_n > 0, \left(k = 2, 3, \cdots, n\right) \tag{1.3}$$

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or equivalent form of (1.3)

$$0 \le \sum_{i=1}^{m} t_i \le \sum_{i=1}^{n} t_i = T_n, \quad T_n > 0 \Big(m = 1, 2, \cdots, n-1 \Big), \tag{1.4}$$

then we have

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le \frac{1}{T_n}\sum_{i=1}^{n} t_i f\left(x_i\right). \tag{1.5}$$

Corollary 1.2. *Let f be a convex function on $\mathbb{I}$ containing x_i, $t_i \in \mathbb{R}/\{0\}(i = 1, 2, \cdots, n; n \ge 2)$. And let $x_1 = min\{x_i | 1 \le i \le n\}$, $x_n = max\{x_i | 1 \le i \le n\}$. If*

$$\sum_{i=2}^{n-1} |t_i| \le \min\{t_1, t_n\}, \quad \sum_{i=1}^{n} t_i = T_n > 0, \quad \left(\sum_{i=2}^{2-1} |t_i| = 0\right), \tag{1.6}$$

then we also obtain (1.5).

Remark If we choose $\frac{t_i}{T_n} = 1 - \omega_i (i = 1, n)$ and $\frac{t_i}{T_n} = -\omega_i (i = 2, 3, \cdots, n-1)$ and $x_1 \le x_2 \le \cdots \le x_n$ in Corollary 1.2, then inequality (1.5) reduces to inequality (1.2).

Towards proving Theorem 1.1 we shall need the following lemma [4]:

Lemma 1.3. *Let f be a convex function on a closed interval $[a, b](a < b)$. Then f'_- and f'_+ exist in (a, b). For any $x_1, x_2 \in (a, b)$, $x_1 < x_2$, we have*

$$f'_-(x_1) \le f'_+(x_1) \le f'_-(x_2) \le f'_+(x_2); \tag{1.7}$$

For any $x_1, x_2 \in [a, b]$, $x_1 < x_2$, then $\exists \xi \in (x_1, x_2)$ such that

$$f'_-(\xi) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le f'_+(\xi). \tag{1.8}$$

2 The Proof of Theorem and Corollary

Proof of Theorem 1.1. Let $x_0 = T_n^{-1} \sum\limits_{i=1}^{n} t_i x_i$. We write

$$T^{(k)} = T_n^{-1} \sum_{i=k}^{n} t_i \qquad (k = 2, 3, \cdots, n)$$

and

$$T_{(m)} = T_n^{-1} \sum_{i=1}^{m} t_i \qquad (m = 1, 2, \cdots, n-1).$$

Using (1.3) and (1.4), we get $0 \le T^{(k)} \le 1 (k = 2, 3, \cdots, n)$ and $0 \le T_{(m)} \le 1 (m = 1, 2, \cdots, n-1)$.

1. When $x_1 \le x_2 \le \cdots \le x_n$, we have

$$x_1 \le x_1 T^{-1} \sum_{i=1}^{n} t_i + \sum_{k=2}^{n} T^{(k)} \left(x_k - x_{k-1}\right) = x_0 = \sum_{m=1}^{n-1} T_{(m)} \left(x_m - x_{m+1}\right) + x_n T^{-1} \sum_{i=1}^{n} t_i \le x_n. \tag{2.1}$$

Case 1. $x_1 = x_0$. From (2.1), $T^{(k)} \geq 0$ and $x_k - x_{k-1} \geq 0$ we have $T^{(k)}(x_k - x_{k-1}) = 0$ i. e. $T^{(k)} = 0$ or $x_k = x_{k-1}$, $(k = 2, 3, \cdots, n)$. Then we get

$$T^{(k)}(f(x_k) - f(x_{k-1})) = 0, \qquad (k = 2, 3, \cdots, n). \tag{2.2}$$

Using (2.2), we have

$$\begin{aligned}
& T_n^{-1}\sum_{i=1}^n t_i f(x_i) - f\left(T_n^{-1}\sum_{i=1}^n t_i x_i\right) \\
= \;& f(x_1)T^{-1}\sum_{i=1}^n t_i + \sum_{k=2}^n T^{(k)}(f(x_k) - f(x_{k-1})) - f(x_0) \\
= \;& f(x_1) - f(x_0) = 0,
\end{aligned}$$

which is equality in (1.5).

Case 2. $x_n = x_0$. By same arguments of proof for case 1, we can also get equality in (1.5).

Case 3. $x_p = x_0 (2 \leq p \leq n-1)$. Using lemma 1.3 we have

$$\begin{aligned}
& T_n^{-1}\sum_{i=1}^n t_i f(x_i) - f\left(T^{-1}\sum_{i=1}^n t_i x_i\right) \\
= \;& f(x_1) - f(x_p) + \sum_{k=2}^n T^{(k)}(f(x_k) - f(x_{k-1})) \\
= \;& \sum_{k=2}^p \left(T^{(k)} - 1\right)(f(x_k) - f(x_{k-1})) + \sum_{k=p+1}^n T^{(k)}(f(x_k) - f(x_{k-1})) \\
\geq \;& \sum_{k=2}^p \left(T^{(k)} - 1\right)(x_k - x_{k-1}) f'_+(\xi_k) \\
& + \sum_{k=p+1}^n T^{(k)}(x_k - x_{k-1}) f'_-(\xi_k), \quad where \quad \xi_k \in (x_{k-1}, x_k), \quad by \quad (1.8), \\
\geq \;& \sum_{k=2}^p \left(T^{(k)} - 1\right)(x_k - x_{k-1}) f'_+(\xi_p), \qquad by \quad (1.7), \\
& + \sum_{k=p+1}^n T^{(k)}(x_k - x_{k-1}) f'_-(\xi_k) \\
= \;& \left[\sum_{k=2}^p T^{(k)}(x_k - x_{k-1}) - (x_0 - x_1)\right] f'_+(\xi_p) + \sum_{k=p+1}^n T^{(k)}(x_k - x_{k-1}) f'_-(\xi_k) \\
= \;& \left[\sum_{k=2}^p T^{(k)}(x_k - x_{k-1}) - \sum_{k=2}^n T^{(k)}(x_k - x_{k-1})\right] f'_+(\xi_p), \qquad by \quad (2.1), \\
& + \sum_{k=p+1}^n T^{(k)}(x_k - x_{k-1}) f'_-(\xi_k) \\
= \;& \sum_{k=p+1}^n T^{(k)}(x_k - x_{k-1}) \left(f'_-(\xi_k) - f'_+(\xi_p)\right) \geq 0, \qquad by \quad (1.7),
\end{aligned}$$

which is (1.5).

Case 4. $x_{p-1} < x_0 < x_p (2 \le p \le n)$. When $2 \le p \le n-1$, using lemma 1.3 we have

$$\begin{aligned}
&T_n^{-1}\sum_{i=1}^{n} t_i f(x_i) - f\left(T^{-1}\sum_{i=1}^{n} t_i x_i\right)\\
&= f(x_1) - f(x_0) + \sum_{k=2}^{n} T^{(k)}\left(f(x_k) - f(x_{k-1})\right)\\
&= \sum_{k=2}^{p-1}\left(T^{(k)} - 1\right)\left(f(x_k) - f(x_{k-1})\right) + \left(T^{(p)} - 1\right)\left(f(x_0) - f(x_{p-1})\right)\\
&\quad + T^{(p)}\left(f(x_p) - f(x_0)\right) + \sum_{k=p+1}^{n} T^{(k)}\left(f(x_k) - f(x_{k-1})\right)\\
&\ge \sum_{k=2}^{p-1}\left(T^{(k)} - 1\right)\left(x_k - x_{k-1}\right) f_+^{'}\left(\xi_k\right), \qquad \xi_k \in \left(x_k - x_{k-1}\right),\\
&\quad + \left(T^{(p)} - 1\right)\left(x_0 - x_{p-1}\right) f_+^{'}\left(\xi_p\right), \qquad \xi_p \in \left(x_0, x_{p-1}\right),\\
&\quad + T^{(p)}\left(x_p - x_0\right) f_-^{'}\left(\xi_p^{'}\right), \qquad \xi_p^{'} \in \left(x_p, x_0\right),\\
&\quad + \sum_{k=p+1}^{n} T^{(k)}\left(x_k - x_{k-1}\right) f_-^{'}\left(\xi_k\right), \qquad \xi_k \in \left(x_k - x_{k-1}\right), \qquad \textit{by} \quad (1.8),\\
&\ge \sum_{k=2}^{p-1}\left(T^{(k)} - 1\right)\left(x_k - x_{k-1}\right) f_+^{'}\left(\xi_p\right) + \left(T^{(p)} - 1\right)\left(x_0 - x_{p-1}\right) f_+^{'}\left(\xi_p\right)\\
&\quad + T^{(p)}\left(x_p - x_0\right) f_+^{'}\left(\xi_p\right) + \sum_{k=p+1}^{n} T^{(k)}\left(x_k - x_{k-1}\right) f_-^{'}\left(\xi_k\right), \qquad \textit{by} \quad (1.7),\\
&= \sum_{k=2}^{p-1} T^{(k)}\left(x_k - x_{k-1}\right) f_+^{'}\left(\xi_p\right) - \left(x_0 - x_1\right) f_+^{'}\left(\xi_p\right) + T^{(p)}\left(x_0 - x_{p-1}\right) f_+^{'}\left(\xi_p\right)\\
&\quad + T^{(p)}\left(x_p - x_0\right) f_+^{'}\left(\xi_p\right) + \sum_{k=p+1}^{n} T^{(k)}\left(x_k - x_{k-1}\right) f_-^{'}\left(\xi_k\right)\\
&= \left[\sum_{k=2}^{p} T^{(k)}\left(x_k - x_{k-1}\right) - \sum_{k=2}^{n} T^{(k)}\left(x_k - x_{k-1}\right)\right] f_+^{'}\left(\xi_p\right), \qquad \textit{by} \quad (2.1),\\
&\quad + \sum_{k=p+1}^{n} T^{(k)}\left(x_k - x_{k-1}\right) f_-^{'}\left(\xi_k\right)\\
&= \sum_{k=p+1}^{n} T^{(k)}\left(x_k - x_{k-1}\right)\left(f_-^{'}\left(\xi_k\right) - f_+^{'}\left(\xi_p\right)\right) \ge 0, \qquad \textit{by} \quad (1.7),
\end{aligned} \tag{2.3}$$

which is (1.5).

When $p = n$ i. e. $x_{n-1} < x_0 < x_n$, We agree

$$\sum_{k=n+1}^{n} \cdots = 0. \tag{2.4}$$

In (2.3), taking $p = n$ and using (2.4), we also get (1.5).

2. When $x_1 \geq x_2 \geq \cdots \geq x_n$, we have

$$x_1 \geq x_1 T^{-1} \sum_{i=1}^{n} t_i + \sum_{k=2}^{n} T^{(k)} (x_k - x_{k-1}) = x_0 = \sum_{m=1}^{n-1} T_{(m)} (x_m - x_{m+1}) + x_n T^{-1} \sum_{i=1}^{n} t_i \geq x_n \tag{2.5}$$

and

$$\begin{aligned} & T_n^{-1} \sum_{i=1}^{n} t_i f(x_i) - f\left(T_n^{-1} \sum_{i=1}^{n} t_i x_i\right) \\ = & \sum_{m=1}^{n-1} T_{(m)} \left(f(x_m) - f(x_{m+1})\right) + f(x_n) - f(x_0). \end{aligned} \tag{2.6}$$

Using (2.5), (2.6) and the same arguments of proof for $x_1 \leq x_2 \leq \cdots \leq x_n$, we also obtain (1.5).

The proof of Theorem 1.1 is completed.

Proof of Corollary 1.2. Case 1. $n = 2$. From (1.6), we have $t_1 > 0$ and $t_n = t_2 > 0$. By (1.1), we get (1.5).

Case 2. $n > 2$. Let $\{m_2, m_3, \cdots, m_{n-1}\}$ be a permutation of $\{2, 3, \cdots, n-1\}$ so that $x_1 \leq x_{m_2} \leq \cdots \leq x_{m_k} \leq x_{m_{k+1}} \leq \cdots \leq x_{m_{n-1}} \leq x_n$. Using (1.6), we get

$$0 \leq \sum_{i=k}^{n-1} t_{m_i} + t_n \leq t_1 + \sum_{i=2}^{n-1} t_{m_i} + t_n = T_n, \quad T_n > 0, \Big(k = 2, 3, \cdots, n-1\Big). \tag{2.7}$$

Using (2.7) and Theorem 1.1, we obtain Corollary 1.2.

The proof of Corollary 1.2 is completed.

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