ON GENERALIZATIONS AND REFINEMENTS OF JORDAN TYPE INEQUALITY

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Abstract. In this paper, we give some refinements and generalizations of Jordan's inequality, we also improve some known results.

1. Introduction and Lemmas

The following inequality is well-known
Assume $0 < |\,x\,| \leq \frac{\pi}{2}$, *then*

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1. \tag{1}$$

It is called Jordan's inequality (see [1]). Jordan's inequality is an important inequality in calculus and trigonometry, It has broad applications in theory of limit, There are many generalizations and refinements that had been published in recent years (see [2] [3] [4] [5]).

In this paper, we give some new generalizations and refinements of Jordan's Inequality. Firstly, we prove following Lemmas.

Lemma 1. *Assume* $0 < \beta < \alpha$, *then the function* $\varphi(x) = \frac{\sin \alpha x}{\sin \beta x}$ *is decreasing on interval* $(0,\ \frac{\pi}{\alpha})$, *and* $\phi(x) = \frac{\tan \alpha x}{\tan \beta x}$ *is increasing on interval* $(0,\ \frac{\pi}{2\alpha})$.

Proof. We note that φ is a differentiable function with

$$\varphi'(x) = \frac{\alpha \cos \alpha x \sin \beta x - \beta \cos \beta x \sin \alpha x}{\sin^2 \beta x}, \quad x \in (0,\ \frac{\pi}{\alpha}).$$

Let $g(x) = \alpha \cos \alpha x \sin \beta x - \beta \cos \beta x \sin \alpha x$, then $g'(x) = (\beta^2 - \alpha^2) \sin \alpha x \sin \beta x$.
Since $0 < \beta < \alpha$, $0 < x < \frac{\pi}{\alpha}$, we have $g'(x) < 0$, $x \in (0,\ \frac{\pi}{\alpha})$.
Consequently, g is decreasing on interval $(0,\ \frac{\pi}{\alpha})$, and then we have
$g(x) < g(0) = 0$ for any $x \in (0,\ \frac{\pi}{\alpha})$,
we deduce that $\varphi'(x) < 0$ for any $x \in (0,\ \frac{\pi}{\alpha})$, it shows that φ is decreasing on interval $(0,\ \frac{\pi}{\alpha})$.

We note that ϕ is also a differentiable function with

$$\phi'(x) = \frac{1}{2} \sec^2 \alpha x \csc^2 \beta x (\alpha \sin 2\beta x - \beta \sin 2\alpha x), \quad x \in (0,\ \frac{\pi}{2\alpha}).$$

Let $u(x) = \alpha \sin 2\beta x - \beta \sin 2\alpha x$, then $u'(x) = 2\alpha\beta\,(\cos 2\beta x - \cos 2\alpha x)$.
Since $0 < \beta < \alpha$, $0 < x < \frac{\pi}{2\alpha}$, we have $u'(x) > 0$, $x \in (0,\ \frac{\pi}{2\alpha})$.

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Consequently, u is increasing on interval $(0,\ \frac{\pi}{2\alpha})$, and then we have
$u(x) > u(0) = 0$ for any $x \in (0,\ \frac{\pi}{2\alpha})$,
we deduce that $\phi'(x) > 0$ for any $x \in (0,\ \frac{\pi}{2\alpha})$, it shows that ϕ is increasing on interval $(0,\ \frac{\pi}{2\alpha})$.

The proof is complete. □

Lemma 2. *Assume* $0 < \beta < \alpha$, $0 < |\alpha x| \leq \frac{\pi}{2}$, *then*

$$2\left|\frac{x}{\pi}\right| \leq \left|(\sin\beta x)/(\alpha\sin\frac{\pi\beta}{2\alpha})\right| \leq \left|\frac{\sin\alpha x}{\alpha}\right| < \left|\frac{\sin\beta x}{\beta}\right| < |\,x\,|. \tag{2}$$

Proof. Case 1, when $0 < x \leq \frac{\pi}{2\alpha}$. By Lemma 1, we see that $\varphi(x) = \frac{\sin\alpha x}{\sin\beta x}$ is decreasing on interval $(0,\ \frac{\pi}{2\alpha})$.
Since

$$\lim_{x\to0^+}\frac{\sin\alpha x}{\sin\beta x} = \lim_{x\to0^+}\frac{\alpha}{\beta}\cdot\frac{\sin\alpha x}{\alpha x}\cdot\frac{\beta x}{\sin\beta x} = \frac{\alpha}{\beta}, \quad \varphi(\ \frac{\pi}{2\alpha}\ ) = \csc\frac{\pi\beta}{2\alpha},$$

thus

$$\frac{\alpha}{\beta} > \frac{\sin\alpha x}{\sin\beta x} \geq \csc\frac{\pi\beta}{2\alpha} \tag{3}$$

for any $0 < \beta < \alpha$ and $0 < x \leq \frac{\pi}{2\alpha}$.

Since $0 < x \leq \frac{\pi}{2\alpha}$, $0 < \beta < \pi/(2(\frac{\pi}{2\alpha}))$, by inequality (3), we obtain

$$(\frac{\pi}{2\alpha})/x \geq (\sin\frac{\pi}{2\alpha}\beta)/(\sin x\beta),$$

that is

$$\frac{2x}{\pi} \leq (\sin\beta x)/(\alpha\sin\frac{\pi\beta}{2\alpha}). \tag{4}$$

Using Jordan's inequality, we have

$$\frac{\sin\beta x}{\beta} < x. \tag{5}$$

Combining inequalities (3),(4),(5), we get

$$0 < \frac{2x}{\pi} \leq (\sin\beta x)/(\alpha\sin\frac{\pi\beta}{2\alpha}) \leq \frac{\sin\alpha x}{\alpha} < \frac{\sin\beta x}{\beta} < x, \tag{6}$$

that is

$$2\left|\frac{x}{\pi}\right| \leq \left|(\sin\beta x)/(\alpha\sin\frac{\pi\beta}{2\alpha})\right| \leq \left|\frac{\sin\alpha x}{\alpha}\right| < \left|\frac{\sin\beta x}{\beta}\right| < |\,x\,|.$$

Case 2, when $-\frac{\pi}{2\alpha} \leq x < 0$, then $0 < -x \leq \frac{\pi}{2\alpha}$. By the result of *Case 1*, we have

$$2\left|\frac{-x}{\pi}\right| \leq \left|(\sin(-\beta x))/(\alpha\sin\frac{\pi\beta}{2\alpha})\right| \leq \left|\frac{\sin(-\alpha x)}{\alpha}\right| < \left|\frac{\sin(-\beta x)}{\beta}\right| < |-x\,|.$$

that is

$$2\left|\frac{x}{\pi}\right| \leq \left|(\sin\beta x)/(\alpha\sin\frac{\pi\beta}{2\alpha})\right| \leq \left|\frac{\sin\alpha x}{\alpha}\right| < \left|\frac{\sin\beta x}{\beta}\right| < |\,x\,|.$$

The proof is complete. □

Lemma 3. *The function* $f(x) = \ln \frac{\sin x}{x}$ *is concave on interval* $(0,\ \frac{\pi}{2})$*, and* $g(x) = \ln \frac{\tan x}{x}$ *is convex on interval* $(0,\ \frac{\pi}{2})$.

Proof. f is a second order differentiable function with

$$f''(x) = \frac{\csc^2 x}{x^2}(\sin^2 x - x^2), \quad x \in (0,\ \frac{\pi}{2}).$$

Using Jordan's inequality, we get $f''(x) < 0$ for any $x \in (0,\ \frac{\pi}{2})$. Therefore f is concave on interval $(0,\ \frac{\pi}{2})$.

By g is also a second order differentiable function with

$$g''(x) = \frac{\csc^2 2x}{x^2}(\sin^2 2x - 4x^2 \cos 2x), \quad x \in (0,\ \frac{\pi}{2}).$$

Firstly, suppose $\frac{\pi}{4} \le x < \frac{\pi}{2}$, it is easy to see that $g''(x) > 0$.

Secondly, consider the case of $0 < x < \frac{\pi}{4}$.

Now

$$g''(x) = \frac{\csc^2 2x}{x^2}(\sin^2 2x - 4x^2 \cos 2x) = \frac{\csc^2 2x \cos 2x}{x^2}(\sin 2x \tan 2x - 4x^2).$$

Let $q(x) = \sin 2x \tan 2x - 4x^2$, we have

$$q'(x) = 2\cos 2x \tan 2x + 2\sin 2x \sec^2 2x - 8x,$$
$$q''(x) = 4\sec 2x(\cos 2x - 1)^2 + 8\sin 2x \tan 2x \sec^2 2x > 0, \quad x \in (0,\ \frac{\pi}{4}),$$

Hence $q'(x)$ is increasing on interval $(0,\ \frac{\pi}{4})$, and then we have $q'(x) > q'(0) = 0$ for any $x \in (0,\ \frac{\pi}{4})$.

Thus $q(x)$ is increasing on interval $(0,\ \frac{\pi}{4})$, further we get $q(x) > q(0) = 0$ for any $x \in (0,\ \frac{\pi}{4})$.

Therefore $g''(x) > 0$ for any $x \in (0,\ \frac{\pi}{4})$, actually $g''(x) > 0$ for any $x \in (0,\ \frac{\pi}{2})$.

It shows that g is convex on interval $(0,\ \frac{\pi}{2})$.

The proof is complete. □

2. Main Results and Their Proof

Theorem 1. *Assume* $0 < |\,y\,| < |\,x\,|$, $0 < |\lambda x| \le \frac{\pi}{2}$, *then the following inequality holds*

$$2\left|\frac{\lambda}{\pi}\right| \le \left|(\sin \lambda y)/(x \sin \frac{\pi y}{2x})\right| \le \left|\frac{\sin \lambda x}{x}\right| < \left|\frac{\sin \lambda y}{y}\right| < |\,\lambda\,|. \tag{7}$$

Proof. By hypothesis, we have

$$0 < |\,y\,| < |\,x\,|, \quad 0 < |\,|x|\,\lambda\,| \le \frac{\pi}{2},$$

Using Lemma 2, we get

$$2\left|\frac{\lambda}{\pi}\right| \le \left|(\sin \lambda\,|y|)/(|x| \sin \frac{\pi}{2}\left|\frac{y}{x}\right|)\right| \le \left|\frac{\sin(\lambda\,|\,x\,|)}{|\,x\,|}\right| < \left|\frac{\sin(\lambda\,|\,y\,|)}{|\,y\,|}\right| < |\,\lambda\,|.$$

By identical equation

$$\left|(\sin \lambda\,|y|)/(|x| \sin \frac{\pi}{2}\left|\frac{y}{x}\right|)\right| = \left|(\sin \lambda y)/(x \sin \frac{\pi y}{2x})\right|,$$

$$\left|\frac{\sin(\lambda\,|\,x\,|\,)}{|\,x\,|}\right| = \left|\frac{\sin\lambda x}{x}\right|, \quad \left|\frac{\sin(\lambda\,|\,y\,|\,)}{|\,y\,|}\right| = \left|\frac{\sin\lambda y}{y}\right|,$$

we immediately obtain inequality (7). The proof of Theorem 1 is complete. □

From Theorem 1 we get

Corollary 1. *Assume* $0 < |\,\lambda x\,| \le \frac{\pi}{2}$*, then the following inequality holds*

$$2\left|\frac{\lambda}{\pi}\right| \le \left|\frac{\sin\lambda x}{x}\right| < |\,\lambda\,|\,. \tag{8}$$

Remark. If we choose $\lambda = 1$ in Corollary 1, inequality (8) become Jordan's inequality.

Corollary 2. *Assume* $0 < |\,y\,| < |\,x\,| \le \frac{\pi}{2}$*, then the following inequality holds*

$$\frac{2}{\pi} \le (\sin y)/(x\sin\frac{\pi y}{2x}) \le \frac{\sin x}{x} < \frac{\sin y}{y} < 1. \tag{9}$$

Proof. Let $\lambda = 1$ in Theorem 1, we have

$$\frac{2}{\pi} \le \left|(\sin y)/(x\sin\frac{\pi y}{2x})\right| \le \left|\frac{\sin x}{x}\right| < \left|\frac{\sin y}{y}\right| < 1 \tag{10}$$

for any $0 < |\,y\,| < |\,x\,| \le \frac{\pi}{2}$,

By inequality (10) and since

$$\frac{\sin x}{x} > 0,\ \frac{\sin y}{y} > 0,\quad (\sin y)/(x\sin\frac{\pi y}{2x}) = \frac{\pi}{2}\cdot\frac{\sin y}{y}\cdot((\frac{\pi y}{2x})/(\sin\frac{\pi y}{2x})) > 0,$$

we obtain inequality (9) □

Remark. The inequality (9) is a refinement of Jordan's inequality .

Theorem 2. *Assume* $0 < \beta < \alpha$*,* $0 < |\alpha x| < \frac{\pi}{2}$*, then the following inequality holds*

$$|\tan\alpha x\,| > \frac{\alpha}{\beta}\,|\tan\beta x\,| > \alpha\,|\,x\,| > \frac{\alpha}{\beta}\,|\sin\beta x\,| > |\sin\alpha x\,| > \csc\frac{\pi\beta}{2\alpha}\,|\sin\beta x\,|\,. \tag{11}$$

Proof. From Lemma 2, we get

$$\frac{\alpha}{\beta}\,|\sin\beta x\,| > |\sin\alpha x\,| > \csc\frac{\pi\beta}{2\alpha}\,|\sin\beta x\,|\,. \tag{12}$$

Firstly, we prove following inequality

$$|\tan\alpha x\,| > \frac{\alpha}{\beta}\,|\tan\beta x\,| \tag{13}$$

for any $0 < |\,\alpha x\,| < \frac{\pi}{2}$.

Case 1, when $0 < x \le \frac{\pi}{2\alpha}$. By Lemma 1, we see that $\phi(x) = \frac{\tan\alpha x}{\tan\beta x}$ is increasing on interval $(0,\ \frac{\pi}{2\alpha})$.

Since

$$\lim_{x\to 0^+}\frac{\tan\alpha x}{\tan\beta x} = \lim_{x\to 0^+}\frac{\alpha}{\beta}\cdot\frac{\sin\alpha x}{\alpha x}\cdot\frac{\beta x}{\sin\beta x}\cdot\frac{\cos\beta x}{\cos\alpha x} = \frac{\alpha}{\beta},$$

therefore $\frac{\tan\alpha x}{\tan\beta x} > \frac{\alpha}{\beta}$, it yields that $\tan\alpha x > \frac{\alpha}{\beta}\tan\beta x$ for any $0 < x < \frac{\pi}{2\alpha}$,

By
$$\tan\alpha x = |\tan\alpha x|\,,\ \tan\beta x = |\tan\beta x|\,,$$
we get inequality (13)

Case 2, when $-\frac{\pi}{2\alpha} < x < 0$, then $0 < -x < \frac{\pi}{2\alpha}$. By the result of *Case 1*, we have
$$|\tan(-\alpha x)| > \frac{\alpha}{\beta}|\tan(-\beta x)|\,,$$
which proves inequality (13).

By the well-known inequality (see [6])
$$|\tan x| > |x| > |\sin x| \tag{14}$$
for any $0 < |x| < \frac{\pi}{2}$, we get
$$\frac{\alpha}{\beta}|\tan\beta x| > \frac{\alpha}{\beta}|\beta x| = \alpha|x|\,,\quad \frac{\alpha}{\beta}|\sin\beta x| < \frac{\alpha}{\beta}|\beta x| = \alpha|x|\,,$$
Combining inequalities (12),(13) and above inequalities, we immediately obtain (11). The proof of Theorem 2 is complete. □

Let $\alpha = 1$ in Theorem 2, we obtain

Corollary 3. *Assume $0 < \beta < 1$, $0 < |x| < \frac{\pi}{2}$, then the following inequality holds*
$$|\tan x| > \frac{1}{\beta}|\tan\beta x| > |x| > \frac{1}{\beta}|\sin\beta x| > |\sin x|\,. \tag{15}$$
Remark. The inequality (15) is a refinement of inequality(14).

Corollary 4. *Assume $0 < \beta < \alpha$, $0 < |\alpha x| < \frac{\pi}{2}$, then the following inequality holds*
$$\frac{\tan\alpha x}{\tan\beta x} > \frac{\alpha}{\beta} > \frac{\sin\alpha x}{\sin\beta x} > \csc\frac{\pi\beta}{2\alpha}. \tag{16}$$
Proof. *Case 1*, when $0 < x < \frac{\pi}{2\alpha}$. Then
$$\sin\alpha x > 0,\quad \sin\beta x > 0,\quad \tan\alpha x > 0,\quad \tan\beta x > 0,$$
From Theorem 2, we obtain inequality (16).

Case 2, when $-\frac{\pi}{2\alpha} < x < 0$, then $0 < -x < \frac{\pi}{2\alpha}$, By the result of *Case 1*, we get
$$\frac{\tan(-\alpha x)}{\tan(-\beta x)} > \frac{\alpha}{\beta} > \frac{\sin(-\alpha x)}{\sin(-\beta x)} > \csc\frac{\pi\beta}{2\alpha},$$
that is inequality (16). □

Let $x = 1$ in Corollary 4, we obtain

Corollary 5. *Assume $0 < \beta < \alpha < \frac{\pi}{2}$, then the following inequality holds*
$$\frac{\alpha}{\beta} > \frac{\sin\alpha}{\sin\beta} > \csc\frac{\pi\beta}{2\alpha}. \tag{17}$$
Remark. Note that $\csc\frac{\pi\beta}{2\alpha} = (\sin\frac{\pi\beta}{2\alpha})^{-1} > (\frac{\pi\beta}{2\alpha})^{-1} = \frac{2\alpha}{\pi\beta}$ for any $0 < \beta < \alpha < \frac{\pi}{2}$. Clearly, inequality (17) sharpens following Garnir's inequality(see [1])
$$\frac{\alpha}{\beta} > \frac{\sin\alpha}{\sin\beta} > \frac{2\alpha}{\pi\beta}. \tag{18}$$

Theorem 3. *Assume* $0 < |x_i| < \frac{\pi}{2}$, $i = 1, 2, \cdots, n$, *then the following inequality holds*

$$|\tan x_1 \tan x_2 \cdots \tan x_n| \geq \left| x_1 x_2 \cdots x_n ((\tan \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n}))^n \right|$$

$$> |x_1 x_2 \cdots x_n| > \left| x_1 x_2 \cdots x_n ((\sin \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n}))^n \right|$$

$$\geq |\sin x_1 \sin x_2 \cdots \sin x_n| . \tag{19}$$

Proof. By Lemma 3 and Jensen's inequality(see [7]), we have

$$\frac{1}{n}(\ln \frac{\tan |x_1|}{|x_1|} + \cdots + \ln \frac{\tan |x_n|}{|x_n|}) \geq \ln((\tan \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n})),$$

$$\frac{1}{n}(\ln \frac{\sin |x_1|}{|x_1|} + \cdots + \ln \frac{\sin |x_n|}{|x_n|}) \leq \ln((\sin \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n})),$$

It is obvious that above inequalities are equivalent to following inequalities

$$\tan |x_1| \cdots \tan |x_n| \geq \left| x_1 x_2 \cdots x_n((\tan \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n}))^n \right|,$$

$$\sin |x_1| \cdots \sin |x_n| \leq \left| x_1 x_2 \cdots x_n((\sin \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n}))^n \right|.$$

Now

$$|\tan x_1 \tan x_2 \cdots \tan x_n| = \tan |x_1| \tan |x_2| \cdots \tan |x_n| ,$$

$$|\sin x_1 \sin x_2 \cdots \sin x_n| = \sin |x_1| \sin |x_2| \cdots \sin |x_n| ,$$

$$\left| ((\tan \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n}))^n \right| > 1,$$

$$\left| ((\sin \frac{|x_1| + \cdots + |x_n|}{n})/(\frac{|x_1| + \cdots + |x_n|}{n}))^n \right| < 1.$$

Base on the above results, we immediately obtain inequality (19). The proof of Theorem 3 is complete. □

Remark. Inequality (19) involves n variables $x_1, x_2, \cdots, x_n$. Clearly, it is the refinement and generalization of Jordan's inequality.

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