REFINEMENTS OF THE SCHWARZ AND HEISENBERG INEQUALITIES IN HILBERT SPACES

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Abstract. Some new refinements of the Schwarz inequality in inner product spaces are given. Applications for discrete and integral inequalities including the Heisenberg inequality for vector-valued functions in Hilbert spaces are provided.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications, is the Schwarz inequality

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2, \qquad x,y\in H \tag{1.1}$$

with equality iff x and y are linearly dependent.

In 1966, S. Kurepa [1], established the following refinement of the Schwarz inequality in inner product spaces that generalises de Bruijn's result for sequences of real and complex numbers [2].

Theorem 1. *Let H be a real Hilbert space and $H_{\mathbb{C}}$ the complexification of H. Then for any pair of vectors $a\in H$, $z\in H_{\mathbb{C}}$*

$$|\langle z,a\rangle|^2 \le \frac{1}{2}\|a\|^2\left(\|z\|^2+|\langle z,\bar{z}\rangle|\right)\le \|a\|^2\|z\|^2. \tag{1.2}$$

In 1985, S.S. Dragomir [3, Theorem 2] obtained a different refinement of (1.1), namely:

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex inner product space and $x,y,e\in H$ with $\|e\|=1$. Then we have the inequality*

$$\|x\|\|y\| \ge |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| + |\langle x,e\rangle\langle e,y\rangle| \ge |\langle x,y\rangle|. \tag{1.3}$$

In the same paper [3, Theorem 3], a further generalisation for orthonormal families has been given (see also [4, Theorem 3]).

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Theorem 3. *Let $\{e_i\}_{i\in H}$ be an orthonormal family in the Hilbert space H. Then for any $x, y \in H$*

$$\|x\| \|y\| \geq \left| \langle x, y\rangle - \sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle \right| + \sum_{i\in I} |\langle x, e_i\rangle \langle e_i, y\rangle| \tag{1.4}$$
$$\geq \left| \langle x, y\rangle - \sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle \right| + \left| \sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle \right|$$
$$\geq |\langle x, y\rangle| .$$

The inequality (1.3) has also been obtained in [4] as a particular case of the following result.

Theorem 4. *Let $x, y, a, b \in H$ be such that*

$$\|a\|^2 \leq 2 \operatorname{Re} \langle x, a\rangle , \qquad \|b\|^2 \leq 2 \operatorname{Re} \langle y, b\rangle .$$

Then we have:

$$\|x\| \|y\| \geq \left(2 \operatorname{Re} \langle x, a\rangle - \|a\|^2\right)^{\frac{1}{2}} \left(2 \operatorname{Re} \langle y, b\rangle - \|b\|^2\right)^{\frac{1}{2}} + |\langle x, y\rangle - \langle x, b\rangle - \langle a, y\rangle + \langle a, b\rangle| . \tag{1.5}$$

Another refinement of the Schwarz inequality for orthornormal vectors in inner product spaces has been obtained by S.S. Dragomir and J. Sándor in [5, Theorem 5].

Theorem 5. *Let $\{e_i\}_{i\in H}$ be orthornormal vectors in the inner product space $(H; \langle\cdot, \cdot\rangle)$. Then*

$$\|x\| \|y\| - |\langle x, y\rangle| \tag{1.6}$$
$$\geq \left(\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \sum_{i=1}^{n} |\langle y, e_i\rangle|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{n} \langle x, e_i\rangle \langle e_i, y\rangle \right| \geq 0$$

and

$$\|x\| \|y\| - \operatorname{Re} \langle x, y\rangle \tag{1.7}$$
$$\geq \left(\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \sum_{i=1}^{n} |\langle y, e_i\rangle|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{n} \operatorname{Re} \left[\langle x, e_i\rangle \langle e_i, y\rangle\right] \geq 0.$$

For some properties of superadditivity, monotonicity, strong superadditivity and strong monotonicity of Schwarz's inequality, see [6]. Here we note only the following refinements of the Schwarz inequality in its different variants for linear operators [6]:

a) *Let H be a Hilbert space and $A, B : H \to H$ two selfadjoint linear operators with $A \geq B \geq 0$, then we have the inequality*

$$\langle Ax, x\rangle^{\frac{1}{2}} \langle Ay, y\rangle^{\frac{1}{2}} - |\langle Ax, y\rangle| \geq \langle Bx, x\rangle^{\frac{1}{2}} \langle By, y\rangle^{\frac{1}{2}} - |\langle Bx, y\rangle| \geq 0 \tag{1.8}$$

and

$$\langle Ax, x\rangle \langle Ay, y\rangle - |\langle Ax, y\rangle|^2 \geq \langle Bx, x\rangle \langle By, y\rangle - |\langle Bx, y\rangle|^2 \geq 0 \tag{1.9}$$

for any $x, y \in H$.

b) *Let $A : H \to H$ be a bounded linear operator on H and let $\|A\| = \sup\{\|Ax\|, \|x\| = 1\}$ the norm of A. Then one has the inequalities*

$$\|A\|^2 (\|x\| \|y\| - |\langle x, y\rangle|) \geq \|Ax\| \|Ay\| - |\langle Ax, Ay\rangle| \geq 0 \tag{1.10}$$

and

$$\|A\|^4 \left(\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right) \geq \|Ax\|^2 \|Ay\|^2 - |\langle Ax, Ay\rangle|^2 \geq 0. \tag{1.11}$$

c) *Let $B : H \to H$ be a linear operator with the property that there exists a constant $m > 0$ such that $\|Bx\| \geq m \|x\|$ for any $x \in H$. Then we have the inequality*

$$\|Bx\| \|By\| - |\langle Bx, By\rangle| \geq m^2 (\|x\| \|y\| - |\langle x, y\rangle|) \geq 0 \tag{1.12}$$

and

$$\|Bx\|^2 \|By\|^2 - |\langle Bx, By\rangle|^2 \geq m^4 \left(\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right) \geq 0. \tag{1.13}$$

For other results related to Schwarz's inequality in inner product spaces, see Chapter XX of [8] and the references therein.

Motivated by the results outlined above, it is the aim of this paper to explore other avenues in obtaining new refinements of the celebrated Schwarz inequality. Applications for vector-valued sequences and integrals in Hilbert spaces are mentioned. Refinements of the Heisenberg inequality for vector-valued functions in Hilbert spaces are also given.

2. Some New Refinements

The following result holds.

Theorem 6. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $r_1, r_2 > 0$. If $x, y \in H$ are with the property that*

$$\|x - y\| \geq r_2 \geq r_1 \geq |\|x\| - \|y\||, \tag{2.1}$$

then we have the following refinement of Schwarz's inequality

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \geq \frac{1}{2}\left(r_2^2 - r_1^2\right) (\geq 0). \tag{2.2}$$

The constant $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a larger quantity.*

Proof. From the first inequality in (2.1) we have

$$\|x\|^2 + \|y\|^2 \geq r_2^2 + 2\operatorname{Re}\langle x, y\rangle . \tag{2.3}$$

Subtracting in (2.3) the quantity $2\|x\|\|y\|$, we get

$$(\|x\| - \|y\|)^2 \geq r_2^2 - 2(\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle). \tag{2.4}$$

Since, by the second inequality in (2.1) we have

$$r_1^2 \geq (\|x\| - \|y\|)^2 , \tag{2.5}$$

hence from (2.4) and (2.5) we deduce the desired inequality (2.2).

To prove the sharpness of the constant $\frac{1}{2}$ in (2.2), let us assume that there is a constant $C > 0$ such that

$$\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle \geq C\left(r_2^2 - r_1^2\right), \tag{2.6}$$

provided that x and y satisfy (2.1).

Let $e \in H$ with $\|e\| = 1$ and for $r_2 > r_1 > 0$, define

$$x = \frac{r_2 + r_1}{2} \cdot e \quad \text{and} \quad y = \frac{r_1 - r_2}{2} \cdot e. \tag{2.7}$$

Then

$$\|x - y\| = r_2 \text{ and } |\|x\| - \|y\|| = r_1,$$

showing that the condition (2.1) is fulfilled with equality.

If we replace x and y as defined in (2.7) into the inequality (2.6), then we get

$$\frac{r_2^2 - r_1^2}{2} \geq C\left(r_2^2 - r_1^2\right),$$

which implies that $C \leq \frac{1}{2}$, and the theorem is completely proved. ∎

The following corollary holds.

Corollary 1. *With the assumptions of Theorem 6, we have the inequality:*

$$\|x\| + \|y\| - \frac{\sqrt{2}}{2}\|x + y\| \geq \frac{\sqrt{2}}{2}\sqrt{r_2^2 - r_1^2}. \tag{2.8}$$

Proof. We have, by (2.2), that

$$(\|x\| + \|y\|)^2 - \|x + y\|^2 = 2(\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle) \geq r_2^2 - r_1^2 \geq 0$$

which gives

$$(\|x\| + \|y\|)^2 \geq \|x + y\|^2 + \left(\sqrt{r_2^2 - r_1^2}\right)^2 . \tag{2.9}$$

By making use of the elementary inequality

$$2\left(\alpha^2+\beta^2\right)\geq(\alpha+\beta)^2, \qquad \alpha,\beta\geq 0;$$

we get

$$\|x+y\|^2+\left(\sqrt{r_2^2-r_1^2}\right)^2\geq\frac{1}{2}\left(\|x+y\|+\sqrt{r_2^2-r_1^2}\right)^2. \tag{2.10}$$

Utilising (2.9) and (2.10), we deduce the desired inequality (2.8). ∎

If $(H;\langle\cdot,\cdot\rangle)$ is a Hilbert space and $\{e_i\}_{i\in I}$ is an orthornormal family in H, i.e., we recall that $\langle e_i,e_j\rangle=\delta_{ij}$ for any $i,j\in I$, where δ_{ij} is Kronecker's delta, then we have the following inequality which is well known in the literature as *Bessel's inequality*

$$\sum_{i\in I}|\langle x,e_i\rangle|^2\leq\|x\|^2 \quad \text{for each} \quad x\in H. \tag{2.11}$$

Here, the meaning of the sum is

$$\sum_{i\in I}|\langle x,e_i\rangle|^2=\sup_{F\subset I}\left\{\sum_{i\in F}|\langle x,e_i\rangle|^2,\ F \text{ is a finite part of } I\right\}.$$

The following result providing a refinement of the Bessel inequality (2.11) holds.

Theorem 7. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\{e_i\}_{i\in I}$ an orthornormal family in H. If $x\in H$, $x\neq 0$, and $r_2,r_1>0$ are such that:*

$$\left\|x-\sum_{i\in I}\langle x,e_i\rangle e_i\right\|\geq r_2\geq r_1\geq\|x\|-\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}(\geq 0), \tag{2.12}$$

then we have the inequality

$$\|x\|-\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\geq\frac{1}{2}\cdot\frac{r_2^2-r_1^2}{\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}}(\geq 0). \tag{2.13}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Consider $y:=\sum_{i\in I}\langle x,e_i\rangle e_i$. Obviously, since H is a Hilbert space, $y\in H$. We also note that

$$\|y\|=\left\|\sum_{i\in I}\langle x,e_i\rangle e_i\right\|=\sqrt{\left\|\sum_{i\in I}\langle x,e_i\rangle e_i\right\|^2}=\sqrt{\sum_{i\in I}|\langle x,e_i\rangle|^2},$$

and thus (2.12) is in fact (2.1) of Theorem 6.

Since

$$\begin{aligned}\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle &= \|x\|\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}-\operatorname{Re}\left\langle x,\sum_{i\in I}\langle x,e_i\rangle e_i\right\rangle\\ &= \left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\left[\|x\|-\left(\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{\frac{1}{2}}\right],\end{aligned}$$

hence, by (2.2), we deduce the desired result (2.13).

We will prove the sharpness of the constant for the case of one element, i.e., $I=\{1\}$, $e_1=e\in H$, $\|e\|=1$. For this, assume that there exists a constant $D>0$ such that

$$\|x\|-|\langle x,e\rangle|\geq D\cdot\frac{r_2^2-r_1^2}{|\langle x,e\rangle|} \tag{2.14}$$

provided $x\in H\backslash\{0\}$ satisfies the condition

$$\|x-\langle x,e\rangle e\|\geq r_2\geq r_1\geq\|x\|-|\langle x,e\rangle|. \tag{2.15}$$

Assume that $x=\lambda e+\mu f$ with $e,f\in H$, $\|e\|=\|f\|=1$ and $e\perp f$. We wish to see if there exists positive numbers λ,μ such that

$$\|x-\langle x,e\rangle e\|=r_2>r_1=\|x\|-|\langle x,e\rangle|. \tag{2.16}$$

Since (for $\lambda,\mu>0$)

$$\|x-\langle x,e\rangle e\|=\mu$$

and

$$\|x\|-|\langle x,e\rangle|=\sqrt{\lambda^2+\mu^2}-\lambda$$

hence, by (2.16), we get $\mu=r_2$ and

$$\sqrt{\lambda^2+r_2^2}-\lambda=r_1$$

giving

$$\lambda^2+r_2^2=\lambda^2+2\lambda r_1+r_1^2$$

from where we get

$$\lambda=\frac{r_2^2-r_1^2}{2r_1}>0.$$

With these values for λ and μ, we have

$$\|x\|-|\langle x,e\rangle|=r_1,\qquad |\langle x,e\rangle|=\frac{r_2^2-r_1^2}{2r_1}$$

and thus, from (2.14), we deduce

$$r_1\geq D\cdot\frac{r_2^2-r_1^2}{\frac{r_2^2-r_1^2}{2r_1}},$$

giving $D \leq \frac{1}{2}$. This proves the theorem. ∎

The following corollary is obvious.

Corollary 2. *Let $x, y \in H$ with $\langle x, y\rangle \neq 0$ and $r_2 \geq r_1 > 0$ such that*

$$\left\| \|y\| x - \frac{\langle x, y\rangle}{\|y\|} \cdot y \right\| \geq r_2 \|y\| \geq r_1 \|y\| \geq \|x\| \|y\| - |\langle x, y\rangle| \, (\geq 0) . \tag{2.17}$$

Then we have the following refinement of the Schwarz's inequality:

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \frac{1}{2} \left(r_2^2 - r_1^2\right) \frac{\|y\|^2}{|\langle x, y\rangle|} \, (\geq 0) . \tag{2.18}$$

The constant $\frac{1}{2}$ is best possible.

The following lemma holds.

Lemma 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $R \geq 1$. For $x, y \in H$, the subsequent statements are equivalent:*

(i) *The following refinement of the triangle inequality holds:*

$$\|x\| + \|y\| \geq R \|x + y\| ; \tag{2.19}$$

(ii) *The following refinement of the Schwarz inequality holds:*

$$\|x\| \|y\| - \operatorname{Re} \langle x, y\rangle \geq \frac{1}{2} \left(R^2 - 1\right) \|x + y\|^2 . \tag{2.20}$$

Proof. Taking the square in (2.19), we have

$$2 \|x\| \|y\| \geq \left(R^2 - 1\right) \|x\|^2 + 2R^2 \operatorname{Re} \langle x, y\rangle + \left(R^2 - 1\right) \|y\|^2 . \tag{2.21}$$

Subtracting from both sides of (2.21) the quantity $2 \operatorname{Re} \langle x, y\rangle$, we obtain

$$\begin{aligned} 2 \left(\|x\| \|y\| - \operatorname{Re} \langle x, y\rangle\right) &\geq \left(R^2 - 1\right) \left[\|x\|^2 + 2 \operatorname{Re} \langle x, y\rangle + \|y\|^2\right] \\ &= \left(R^2 - 1\right) \|x + y\|^2 , \end{aligned}$$

which is clearly equivalent to (2.20). ∎

By the use of the above lemma, we may now state the following theorem concerning another refinement of the Schwarz inequality.

Theorem 8. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field and $R \geq 1$, $r \geq 0$. If $x, y \in H$ are such that*

$$\frac{1}{R} \left(\|x\| + \|y\|\right) \geq \|x + y\| \geq r, \tag{2.22}$$

then we have the following refinement of the Schwarz inequality

$$\|x\| \|y\| - \operatorname{Re} \langle x, y\rangle \geq \frac{1}{2} \left(R^2 - 1\right) r^2. \tag{2.23}$$

The constant $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a larger quantity.*

Proof. The inequality (2.23) follows easily from Lemma 1. We need only prove that $\frac{1}{2}$ is the best possible constant in (2.23).

Assume that there exists a $C > 0$ such that

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \geq C\left(R^2 - 1\right) r^2 \tag{2.24}$$

provided x, y, R and r satisfy (2.22).

Consider $r = 1$, $R > 1$ and choose $x = \frac{1-R}{2}e$, $y = \frac{1+R}{2}e$ with $e \in H$, $\|e\| = 1$. Then

$$x + y = e, \quad \frac{\|x\| + \|y\|}{R} = 1$$

and thus (2.22) holds with equality on both sides.

From (2.24), for the above choices, we have $\frac{1}{2}\left(R^2 - 1\right) \geq C\left(R^2 - 1\right)$, which shows that $C \leq \frac{1}{2}$. ∎

Finally, the following result also holds.

Theorem 9. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $r \in (0, 1]$. *For* $x, y \in H$, *the following statements are equivalent:*

(i) *We have the inequality*

$$\left|\|x\| - \|y\|\right| \leq r \|x - y\|; \tag{2.25}$$

(ii) *We have the following refinement of the Schwarz inequality*

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \geq \frac{1}{2}\left(1 - r^2\right) \|x - y\|^2. \tag{2.26}$$

The constant $\frac{1}{2}$ *in (2.26) is best possible.*

Proof. Taking the square in (2.25), we have

$$\|x\|^2 - 2\|x\| \|y\| + \|y\|^2 \leq r^2\left(\|x\|^2 - 2\operatorname{Re}\langle x, y\rangle + \|y\|^2\right)$$

which is clearly equivalent to

$$\left(1 - r^2\right)\left[\|x\|^2 - 2\operatorname{Re}\langle x, y\rangle + \|y\|^2\right] \leq 2\left(\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle\right)$$

or with (2.26).

Now, assume that (2.26) holds with a constant $E > 0$, i.e.,

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \geq E\left(1 - r^2\right) \|x - y\|^2, \tag{2.27}$$

provided (2.25) holds.

Define $x = \frac{r+1}{2}e$, $y = \frac{r-1}{2}e$ with $e \in H$, $\|e\| = 1$. Then

$$\left|\|x\| - \|y\|\right| = r, \quad \|x - y\| = 1$$

showing that (2.25) holds with equality.

If we replace x and y in (2.27), then we get $E\left(1-r^{2}\right)\leq\frac{1}{2}\left(1-r^{2}\right),$ implying that $E\leq\frac{1}{2}$. ∎

3. Discrete Inequalities

Assume that $(K;(\cdot,\cdot))$ is a Hilbert space over the real or complex number field. Assume also that $p_i\geq 0$, $i\in H$ with $\sum_{i=1}^{\infty}p_i=1$ and define

$$\ell_p^2\left(K\right):=\left\{\mathbf{x}:=\left(x_i\right)_{i\in\mathbb{N}}\Big|\, x_i\in\mathbb{K},\ i\in\mathbb{N}\ \text{ and }\ \sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2<\infty\right\}.$$

It is well known that $\ell_p^2\left(K\right)$ endowed with the inner product $\left\langle\cdot,\cdot\right\rangle_p$ defined by

$$\left\langle\mathbf{x},\mathbf{y}\right\rangle_p:=\sum_{i=1}^{\infty}p_i\left(x_i,y_i\right)$$

and generating the norm

$$\left\|\mathbf{x}\right\|_p:=\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

We may state the following discrete inequality improving the Cauchy-Bunyakovsky-Schwarz classical result.

Proposition 1. *Let $(K;(\cdot,\cdot))$ be a Hilbert space and $p_i\geq 0$ $(i\in\mathbb{N})$ with $\sum_{i=1}^{\infty}p_i=1$. Assume that $\mathbf{x},\mathbf{y}\in\ell_p^2\left(K\right)$ and $r_1,r_2>0$ satisfy the condition*

$$\left\|x_i-y_i\right\|\geq r_2\geq r_1\geq\left|\left\|x_i\right\|-\left\|y_i\right\|\right| \tag{3.1}$$

for each $i\in\mathbb{N}$. Then we have the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality

$$\begin{aligned}&\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\sum_{i=1}^{\infty}p_i\operatorname{Re}\left(x_i,y_i\right)\\ \geq\ &\frac{1}{2}\left(r_2^2-r_1^2\right)\geq 0.\end{aligned} \tag{3.2}$$

The constant $\frac{1}{2}$ is best possible.

Proof. From the condition (3.1) we simply deduce

$$\sum_{i=1}^{\infty} p_i \left\| x_i - y_i \right\|^2 \geq r_2^2 \geq r_1^2 \geq \sum_{i=1}^{\infty} p_i \left(\left\| x_i \right\| - \left\| y_i \right\| \right)^2 \geq \left[\left(\sum_{i=1}^{\infty} p_i \left\| x_i \right\|^2 \right)^{\frac{1}{2}} - \left(\sum_{i=1}^{\infty} p_i \left\| y_i \right\|^2 \right)^{\frac{1}{2}} \right]^2 . \tag{3.3}$$

In terms of the norm $\left\| \cdot \right\|_p$, the inequality (3.3) may be written as

$$\left\| \mathbf{x} - \mathbf{y} \right\|_p \geq r_2 \geq r_1 \geq \left| \left\| \mathbf{x} \right\|_p - \left\| \mathbf{y} \right\|_p \right| . \tag{3.4}$$

Utilising Theorem 6 for the Hilbert space $\left(\ell_p^2 (K), \langle \cdot, \cdot \rangle_p \right)$, we deduce the desired inequality (3.2).

For $n = 1$ $(p_1 = 1)$, the inequality (3.2) reduces to (2.2) for which we have shown that $\frac{1}{2}$ is the best possible constant. ∎

By the use of Corollary 1, we may state the following result as well.

Corollary 3. *With the assumptions of Proposition 1, we have the inequality*

$$\left(\sum_{i=1}^{\infty} p_i \left\| x_i \right\|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} p_i \left\| y_i \right\|^2 \right)^{\frac{1}{2}} - \frac{\sqrt{2}}{2} \left(\sum_{i=1}^{\infty} p_i \left\| x_i + y_i \right\|^2 \right)^{\frac{1}{2}} \geq \frac{\sqrt{2}}{2} \sqrt{r_2^2 - r_1^2}. \tag{3.5}$$

The following proposition also holds.

Proposition 2. *Let $(K; (\cdot, \cdot))$ be a Hilbert space and $p_i \geq 0$ $(i \in \mathbb{N})$ with $\sum_{i=1}^{\infty} p_i = 1$. Assume that $\mathbf{x}, \mathbf{y} \in \ell_p^2 (K)$ and $R \geq 1$, $r \geq 0$ satisfy the condition*

$$\frac{1}{R} \left(\left\| x_i \right\| + \left\| y_i \right\| \right) \geq \left\| x_i + y_i \right\| \geq r \tag{3.6}$$

for each $i \in \mathbb{N}$. Then we have the following refinement of the Schwarz inequality

$$\left(\sum_{i=1}^{\infty} p_i \left\| x_i \right\|^2 \sum_{i=1}^{\infty} p_i \left\| y_i \right\|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re} (x_i, y_i) \geq \frac{1}{2} \left(R^2 - 1 \right) r^2. \tag{3.7}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a larger quantity.

Proof. By (3.6) we deduce

$$\frac{1}{R}\left[\sum_{i=1}^{\infty} p_i \left(\|x_i\| + \|y_i\|\right)^2\right]^{\frac{1}{2}} \geq \left(\sum_{i=1}^{\infty} p_i \|x_i + y_i\|^2\right)^{\frac{1}{2}} \geq r. \tag{3.8}$$

By the classical Minkowsky inequality for nonnegative numbers, we have

$$\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} \geq \left[\sum_{i=1}^{\infty} p_i \left(\|x_i\| + \|y_i\|\right)^2\right]^{\frac{1}{2}}, \tag{3.9}$$

and thus, by utilising (3.8) and (3.9), we may state in terms of $\|\cdot\|_p$ the following inequality

$$\frac{1}{R}\left(\|\mathbf{x}\|_p + \|\mathbf{y}\|_p\right) \geq \|\mathbf{x} + \mathbf{y}\|_p \geq r. \tag{3.10}$$

Employing Theorem 8 for the Hilbert space $\ell_p^2(K)$ and the inequality (3.10), we deduce the desired result (3.7).

Since, for $p = 1$, $n = 1$, (3.7) reduced to (2.23) for which we have shown that $\frac{1}{2}$ is the best constant, we conclude that $\frac{1}{2}$ is the best constant in (3.7) as well. ∎

Finally, we may state and prove the following result incorporated in

Proposition 3. *Let $(K; (\cdot, \cdot))$ be a Hilbert space and $p_i \geq 0$ $(i \in \mathbb{N})$ with $\sum_{i=1}^{\infty} p_i = 1$. Assume that $\mathbf{x}, \mathbf{y} \in \ell_p^2(K)$ and $r \in (0, 1]$ such that*

$$\left|\|x_i\| - \|y_i\|\right| \leq r \|x_i - y_i\| \quad \text{for each } i \in \mathbb{N}, \tag{3.11}$$

holds true. Then we have the following refinement of the Schwarz inequality

$$\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}(x_i, y_i) \geq \frac{1}{2}\left(1 - r^2\right) \sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2. \tag{3.12}$$

The constant $\frac{1}{2}$ is best possible in (3.12).

Proof. From (3.11) we have

$$\left[\sum_{i=1}^{\infty} p_i \left(\|x_i\| - \|y_i\|\right)^2\right]^{\frac{1}{2}} \leq r \left[\sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2\right]^{\frac{1}{2}}.$$

Utilising the following elementary result

$$\left|\left(\sum_{i=1}^{\infty}p_i\|x_i\|^2\right)^{\frac{1}{2}}-\left(\sum_{i=1}^{\infty}p_i\|y_i\|^2\right)^{\frac{1}{2}}\right|\leq\left(\sum_{i=1}^{\infty}p_i\left(\|x_i\|-\|y_i\|\right)^2\right)^{\frac{1}{2}},$$

we may state that

$$\left|\|\mathbf{x}\|_p-\|\mathbf{y}\|_p\right|\leq r\|\mathbf{x}-\mathbf{y}\|_p.$$

Now, by making use of Theorem 9, we deduce the desired inequality (3.12) and the fact that $\frac{1}{2}$ is the best possible constant. We omit the details. ∎

4. Integral Inequalities

Assume that $(K;(\cdot,\cdot))$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho:[a,b]\subset\mathbb{R}\to[0,\infty)$ is a Lebesgue integrable function with $\int_a^b\rho(t)\,dt=1$, then we may consider the space $L_\rho^2([a,b];K)$ of all functions $f:[a,b]\to K$, that are Bochner measurable and $\int_a^b\rho(t)\|f(t)\|^2dt<\infty$. It is known that $L_\rho^2([a,b];K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f,g\rangle_\rho:=\int_a^b\rho(t)\,(f(t),g(t))\,dt$$

and generating the norm

$$\|f\|_\rho:=\left(\int_a^b\rho(t)\|f(t)\|^2dt\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

Now we may state and prove the first refinement of the Cauchy-Bunyakovsky-Schwarz integral inequality.

Proposition 4. *Assume that $f,g\in L_\rho^2([a,b];K)$ and $r_2,r_1>0$ satisfy the condition*

$$\|f(t)-g(t)\|\geq r_2\geq r_1\geq\left|\|f(t)\|-\|g(t)\|\right| \tag{4.1}$$

for a.e. $t\in[a,b]$. Then we have the inequality

$$\begin{aligned}&\left(\int_a^b\rho(t)\|f(t)\|^2dt\int_a^b\rho(t)\|g(t)\|^2dt\right)^{\frac{1}{2}}\\&-\int_a^b\rho(t)\operatorname{Re}(f(t),g(t))\,dt\\&\geq\frac{1}{2}\left(r_2^2-r_1^2\right)(\geq 0).\end{aligned} \tag{4.2}$$

The constant $\frac{1}{2}$ is best possible in (4.2).

Proof. Integrating (4.1), we get

$$\left(\int_a^b \rho(t)\left(\|f(t)-g(t)\|\right)^2 dt\right)^{\frac{1}{2}} \geq r_2 \geq r_1 \geq \left(\int_a^b \rho(t)\left(\|f(t)\|-\|g(t)\|\right)^2 dt\right)^{\frac{1}{2}}. \tag{4.3}$$

Utilising the obvious fact

$$\left[\int_a^b \rho(t)\left(\|f(t)\|-\|g(t)\|\right)^2 dt\right]^{\frac{1}{2}} \geq \left|\left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} - \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}}\right|, \tag{4.4}$$

we can state the following inequality in terms of the $\|\cdot\|_\rho$ norm:

$$\|f-g\|_\rho \geq r_2 \geq r_1 \geq \left|\|f\|_\rho - \|g\|_\rho\right|. \tag{4.5}$$

Employing Theorem 6 for the Hilbert space $L^2_\rho([a,b];K)$, we deduce the desired inequality (4.2).

To prove the sharpness of $\frac{1}{2}$ in (4.2), we choose $a=0$, $b=1$, $f(t)=1$, $t\in[0,1]$ and $f(t)=x$, $g(t)=y$, $t\in[a,b]$, $x,y\in K$. Then (4.2) becomes

$$\|x\|\,\|y\| - \operatorname{Re}\langle x,y\rangle \geq \frac{1}{2}\left(r_2^2 - r_1^2\right)$$

provided

$$\|x-y\| \geq r_2 \geq r_1 \geq \left|\|x\|-\|y\|\right|,$$

which, by Theorem 6 has the quantity $\frac{1}{2}$ as the best possible constant. ∎

The following corollary holds.

Corollary 4. *With the assumptions of Proposition 4, we have the inequality*

$$\left(\int_a^b \rho(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} + \left(\int_a^b \rho(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}} - \frac{\sqrt{2}}{2}\left(\int_a^b \rho(t)\|f(t)+g(t)\|^2 dt\right)^{\frac{1}{2}} \geq \frac{\sqrt{2}}{2}\sqrt{r_2^2 - r_1^2}. \tag{4.6}$$

The following two refinements of the Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality also hold.

Proposition 5. *If $f,g\in L_{\rho}^{2}([a,b];K)$ and $R\geq 1,r\geq 0$ satisfy the condition*

$$\frac{1}{R}\left(\left\|f(t)\right\|+\left\|g(t)\right\|\right)\geq\left\|f(t)+g(t)\right\|\geq r \tag{4.7}$$

for a.e. $t\in[a,b]$, then we have the inequality

$$\begin{aligned}&\left(\int_a^b\rho(t)\left\|f(t)\right\|^2dt\int_a^b\rho(t)\left\|g(t)\right\|^2dt\right)^{\frac{1}{2}}\\&-\int_a^b\rho(t)\operatorname{Re}\left(f(t),g(t)\right)dt\\&\geq\frac{1}{2}\left(R^2-1\right)r^2.\end{aligned} \tag{4.8}$$

The constant $\frac{1}{2}$ is best possible in (4.8).

The proof follows by Theorem 8 and we omit the details.

Proposition 6. *If $f,g\in L_{\rho}^{2}([a,b];K)$ and $\zeta\in(0,1]$ satisfy the condition*

$$\left|\left\|f(t)\right\|-\left\|g(t)\right\|\right|\leq\zeta\left\|f(t)-g(t)\right\| \tag{4.9}$$

for a.e. $t\in[a,b]$, then we have the inequality

$$\begin{aligned}&\left(\int_a^b\rho(t)\left\|f(t)\right\|^2dt\int_a^b\rho(t)\left\|g(t)\right\|^2dt\right)^{\frac{1}{2}}\\&-\int_a^b\rho(t)\operatorname{Re}\left(f(t),g(t)\right)dt\\&\geq\frac{1}{2}\left(1-\zeta^2\right)\int_a^b\rho(t)\left\|f(t)-g(t)\right\|^2dt.\end{aligned} \tag{4.10}$$

The constant $\frac{1}{2}$ is best possible in (4.10).

The proof follows by Theorem 9 and we omit the details.

5. Refinements of Heisenberg Inequality

It is well known that if $(H;\langle\cdot,\cdot\rangle)$ is a real or complex Hilbert space and $f:[a,b]\subset\mathbb{R}\rightarrow H$ is an *absolutely continuous vector-valued* function, then f is differentiable almost everywhere on $[a,b]$, the derivative $f':[a,b]\rightarrow H$ is Bochner integrable on $[a,b]$ and

$$f(t)=\int_a^t f'(s)\,ds\qquad\text{for any }\ t\in[a,b]. \tag{5.1}$$

The following theorem provides a version of the Heisenberg inequalities in the general setting of Hilbert spaces.

Theorem 10. *Let $\varphi : [a,b] \to H$ be an absolutely continuous function with the property that $b\left\Vert \varphi(b)\right\Vert^2 = a\left\Vert \varphi(a)\right\Vert^2$. Then we have the inequality:*

$$\left(\int_a^b \left\Vert \varphi(t)\right\Vert^2 dt\right)^2 \le 4\int_a^b t^2 \left\Vert \varphi(t)\right\Vert^2 dt \cdot \int_a^b \left\Vert \varphi'(t)\right\Vert^2 dt. \tag{5.2}$$

The constant 4 is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Integrating by parts, we have successively

$$\begin{aligned} &\int_a^b \left\Vert \varphi(t)\right\Vert^2 dt \\ &= t\left\Vert \varphi(t)\right\Vert^2 \Big|_a^b - \int_a^b t\frac{d}{dt}\left(\left\Vert \varphi(t)\right\Vert^2\right) dt \\ &= b\left\Vert \varphi(b)\right\Vert^2 - a\left\Vert \varphi(a)\right\Vert^2 - \int_a^b t\frac{d}{dt}\left\langle \varphi(t), \varphi(t)\right\rangle dt \\ &= -\int_a^b t\left[\left\langle \varphi'(t), \varphi(t)\right\rangle + \left\langle \varphi(t), \varphi'(t)\right\rangle\right] dt \\ &= -2\int_a^b t \operatorname{Re}\left\langle \varphi'(t), \varphi(t)\right\rangle dt \\ &= 2\int_a^b \operatorname{Re}\left\langle \varphi'(t), (-t)\varphi(t)\right\rangle dt. \end{aligned} \tag{5.3}$$

If we apply the Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_a^b \operatorname{Re}\left\langle g(t), h(t)\right\rangle dt \le \left(\int_a^b \left\Vert g(t)\right\Vert^2 dt \int_a^b \left\Vert h(t)\right\Vert^2 dt\right)^{\frac{1}{2}}$$

for $g(t) = \varphi'(t)$, $h(t) = -t\varphi(t)$, $t \in [a,b]$, then we deduce the desired inequality (4.2).

The fact that 4 is the best constant in (4.2) follows from the fact that in the (CBS) inequality, the case of equality holds iff $g(t) = \lambda h(t)$ for a.e. $t \in [a,b]$ and λ a given scalar in $\mathbb{K}$. We omit the details. ∎

For details on the classical Heisenberg inequality, see, for instance, [7].

Utilising Proposition 4, we can state the following refinement of the Heisenberg inequality obtained above in (5.2):

Proposition 7. *Assume that $\varphi : [a,b] \to H$ is as in the hypothesis of Theorem 10. In addition, if there exists $r_2, r_1 > 0$ so that*

$$\left\Vert \varphi'(t) + t\varphi(t)\right\Vert \ge r_2 \ge r_1 \ge \left|\left\Vert \varphi'(t)\right\Vert - |t|\left\Vert \varphi(t)\right\Vert\right|$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\left(\int_a^b t^2 \|\varphi(t)\|^2 dt \cdot \int_a^b \|\varphi'(t)\|^2 dt\right)^{1/2} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt$$
$$\geq \frac{1}{2}(b-a)\left(r_2^2 - r_1^2\right)(\geq 0).$$

The proof follows by Proposition 4 on choosing $f(t) = \varphi'(t)$, $g(t) = -t\varphi(t)$ and $\rho(t) = \frac{1}{b-a}$, $t \in [a,b]$.

On utilising the Proposition 5 for the same choices of f, g and ρ, we may state the following results as well:

Proposition 8. *Assume that* $\varphi : [a,b] \to H$ *is as in the hypothesis of Theorem 10. In addition, if there exists* $R \geq 1$ *and* $r > 0$ *so that*

$$\frac{1}{R}\left(\|\varphi'(t)\| + |t|\,\|\varphi(t)\|\right) \geq \|\varphi'(t) - t\varphi(t)\| \geq r$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\left(\int_a^b t^2 \|\varphi(t)\|^2 dt \cdot \int_a^b \|\varphi'(t)\|^2 dt\right)^{1/2} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt$$
$$\geq \frac{1}{2}(b-a)\left(R^2 - 1\right)r^2(\geq 0).$$

Finally, we can state

Proposition 9. *Let* $\varphi : [a,b] \to H$ *be as in the hypothesis of Theorem 10. In addition, if there exists* $\zeta \in (0,1]$ *so that*

$$\left|\|\varphi'(t)\| - |t|\,\|\varphi(t)\|\right| \leq \zeta \|\varphi'(t) + t\varphi(t)\|$$

for a.e. $t \in [a,b]$, *then we have the inequality*

$$\left(\int_a^b t^2 \|\varphi(t)\|^2 dt \cdot \int_a^b \|\varphi'(t)\|^2 dt\right)^{1/2} - \frac{1}{2}\int_a^b \|\varphi(t)\|^2 dt$$
$$\geq \frac{1}{2}\left(1 - \zeta^2\right)\int_a^b \|\varphi'(t) + t\varphi(t)\|^2 dt\,(\geq 0).$$

This follows by Proposition 6 and we omit the details.

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