QUADRATIC REVERSES OF THE CONTINUOUS TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN HILBERT SPACES

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Abstract. Some quadratic reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in Hilbert spaces are given. Applications for complex-valued functions are provided as well.

1. Introduction

Let $f:[a,b]\to\mathbb{K}$, $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality is the continuous version of the triangle inequality

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \tag{1.1}$$

and plays a fundamental role in Mathematical Analysis and its applications.

It seems, see [6, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [4]:

$$\cos\theta\int_a^b |f(x)|\,dx \leq \left|\int_a^b f(x)\,dx\right| \tag{1.2}$$

provided

$$|\arg f(x)| \leq \theta, \quad x\in[a,b],$$

where θ is a given angle in $\left(0,\frac{\pi}{2}\right)$.

This integral inequality is the continuous version of a reverse inequality for the generalised triangle inequality

$$\cos\theta\sum_{i=1}^n |z_i| \leq \left|\sum_{i=1}^n z_i\right|, \tag{1.3}$$

provided

$$a-\theta\leq\arg(z_i)\leq a+\theta, \quad \text{for } i\in\{1,\dots,n\},$$

where $a\in\mathbb{R}$ and $\theta\in\left(0,\frac{\pi}{2}\right)$, which, as pointed out in [6, p. 492], was first discovered by M. Petrovich in 1917, [7], and, subsequently rediscovered by other authors, including J. Karamata [4, p. 300 – 301], H.S. Wilf [8], and in an equivalent form by M. Marden [5].

The first to consider the problem for sums in the more general case of Hilbert and Banach spaces, were J.B. Diaz and F.T. Metcalf [1].

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In our previous work [2], we pointed out some continuous versions of Diaz and Metcalf results providing reverses of the generalised triangle inequality in Hilbert spaces.

We mention here some results from [2] which may be compared with the new ones obtained in Sections 2 and 3 below.

Theorem 1. *If $f \in L([a,b];H)$, the space of Bochner integral functions defined on $[a,b]$ and with values in the Hilbert space H, and there exists a constant $K \geq 1$ and a vector $e \in H$, $\|e\| = 1$ such that*

$$\|f(t)\| \leq K \operatorname{Re}\langle f(t), e\rangle \quad \text{for a.e. } t \in [a,b], \tag{1.4}$$

then we have the inequality:

$$\int_a^b \|f(t)\|\,dt \leq K \left\| \int_a^b f(t)\,dt \right\|. \tag{1.5}$$

The case of equality holds in (1.5) if and only if

$$\int_a^b f(t)\,dt = \frac{1}{K}\left(\int_a^b \|f(t)\|\,dt\right) e. \tag{1.6}$$

As particular cases of interest that may be applied in practice, we note the following corollaries established in [2].

Corollary 1. *Let e be a unit vector in the Hilbert space $(H;\langle\cdot,\cdot\rangle)$, $\rho \in (0,1)$ and $f \in L([a,b];H)$ so that*

$$\|f(t) - e\| \leq \rho \quad \text{for a.e. } t \in [a,b]. \tag{1.7}$$

Then we have the inequality

$$\sqrt{1-\rho^2}\int_a^b \|f(t)\|\,dt \leq \left\| \int_a^b f(t)\,dt \right\|, \tag{1.8}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \sqrt{1-\rho^2}\left(\int_a^b \|f(t)\|\,dt\right)\cdot e. \tag{1.9}$$

Corollary 2. *Let e be a unit vector in H and $M \geq m > 0$. If $f \in L([a,b];H)$ is such that*

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \geq 0 \tag{1.10}$$

or, equivalently,

$$\left\| f(t) - \frac{M+m}{2}e \right\| \leq \frac{1}{2}(M-m) \tag{1.11}$$

for a.e. $t \in [a,b]$, then we have the inequality

$$\frac{2\sqrt{mM}}{M+m}\int_a^b \|f(t)\|\,dt \leq \left\| \int_a^b f(t)\,dt \right\|, \tag{1.12}$$

or, equivalently

$$0 \leq \int_a^b \|f(t)\|\,dt - \left\| \int_a^b f(t)\,dt \right\| \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{M+m}\left\| \int_a^b f(t)\,dt \right\|. \tag{1.13}$$

The equality holds in (1.12) (or in the second part of (1.13)) if and only if

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m}\left(\int_a^b \|f(t)\|\,dt\right)e.$$

The case of additive reverse inequalities for the continuous triangle inequality has been considered in [3].

We recall here the following general result.

Theorem 2. *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $k : [a,b] \to [0,\infty)$ a Lebesgue integrable function such that*

$$\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle \le k(t) \quad \text{for a.e. } t \in [a,b], \tag{1.14}$$

then we have the inequality:

$$(0 \le) \int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \le \int_a^b k(t)\,dt. \tag{1.15}$$

The equality holds in (1.15) if and only if

$$\int_a^b \|f(t)\|\,dt \ge \int_a^b k(t)\,dt \tag{1.16}$$

and

$$\int_a^b f(t)\,dt = \left(\int_a^b \|f(t)\|\,dt - \int_a^b k(t)\,dt\right)e. \tag{1.17}$$

This general result has some particular cases of interest that may be easily applied [3].

Corollary 3. *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $\rho \in (0,1)$ such that*

$$\|f(t) - e\| \le \rho \quad \text{for a.e. } t \in [a,b], \tag{1.18}$$

then

$$\begin{aligned} 0 &\le \int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \\ &\le \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle. \end{aligned} \tag{1.19}$$

Corollary 4. *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $M \ge m > 0$ such that either*

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \ge 0 \tag{1.20}$$

or, equivalently,

$$\left\|f(t) - \frac{M+m}{2}e\right\| \le \frac{1}{2}(M-m) \tag{1.21}$$

for a.e. $t \in [a,b]$, *then*

$$0 \leq \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \left\langle \int_a^b f(t) \, dt, e \right\rangle; \tag{1.22}$$

and finally,

Corollary 5. *If* $f \in L([a,b];H)$ *and* $r \in L_2([a,b];H)$, $e \in H$, $\|e\| = 1$ *are such that*

$$\|f(t) - e\| \leq r(t) \quad \text{for a.e. } t \in [a,b], \tag{1.23}$$

then

$$(0 \leq) \int_a^b \|f(t)\| \, dt - \left\| \int_a^b f(t) \, dt \right\| \leq \frac{1}{2} \int_a^b r^2(t) \, dt. \tag{1.24}$$

The main aim of this paper is to point out some quadratic reverses for the continuous triangle inequality, namely, upper bounds for the nonnegative difference

$$\left(\int_a^b \|f(t)\| \, dt \right)^2 - \left\| \int_a^b f(t) \, dt \right\|^2$$

under various assumptions on the functions $f \in L([a,b];H)$.Some related results are also pointed out. Applications for complex-valued functions are provided as well.

2. Quadratic Reverses of the Triangle Inequality

The following lemma holds.

Lemma 1. *Let* $f \in L([a,b];H)$ *be such that there exists a functions* $k : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$, $\Delta := \{(t,s) \,|\, a \leq t \leq s \leq b\}$ *with the property that* $k \in L(\Delta)$ *and*

$$(0 \leq) \|f(t)\| \|f(s)\| - \operatorname{Re} \langle f(t), f(s) \rangle \leq k(t,s), \tag{2.1}$$

for a.e. $(t,s) \in \Delta$. *Then we have the following quadratic reverse of the continuous triangle inequality:*

$$\left(\int_a^b \|f(t)\| \, dt \right)^2 \leq \left\| \int_a^b f(t) \, dt \right\|^2 + 2 \iint_\Delta k(t,s) \, dt ds. \tag{2.2}$$

The case of equality holds in (2.2) if and only if it holds in (2.1) for a.e. $(t,s) \in \Delta$.

Proof. We observe that the following identity holds

$$\begin{aligned}(2.3)\qquad &\left(\int_a^b \|f(t)\|\,dt\right)^2 - \left\|\int_a^b f(t)\,dt\right\|^2\\ &= \int_a^b\int_a^b \|f(t)\|\,\|f(s)\|\,dtds - \left\langle \int_a^b f(t)\,dt, \int_a^b f(s)\,ds\right\rangle\\ &= \int_a^b\int_a^b \|f(t)\|\,\|f(s)\|\,dtds - \int_a^b\int_a^b \operatorname{Re}\langle f(t), f(s)\rangle\,dtds\\ &= \int_a^b\int_a^b \left[\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle\right] dtds := I.\end{aligned}$$

Now, observe that for any $(t,s)\in[a,b]\times[a,b]$, we have

$$\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle = \|f(s)\|\,\|f(t)\| - \operatorname{Re}\langle f(s), f(t)\rangle$$

and thus

$$(2.4)\qquad I = 2\iint_\Delta \left[\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle\right] dtds.$$

Using the assumption (2.1), we deduce

$$\iint_\Delta \left[\|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle\right] dtds \le \iint_\Delta k(t,s)\,dtds,$$

and, by the identities (2.3) and (2.4), we deduce the desired inequality (2.2).

The case of equality is obvious and we omit the details. ∎

Remark 1. *From (2.2) one may deduce a coarser inequality that can be useful in some applications. It is as follows:*

$$(0\le)\int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \le \sqrt{2}\left(\iint_\Delta k(t,s)\,dtds\right)^{\frac{1}{2}}.$$

Remark 2. *If the condition (2.1) is replaced with the following refinement of the Schwarz inequality*

$$(2.5)\qquad (0\le)\,k(t,s) \le \|f(t)\|\,\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle$$

for a.e. $(t,s)\in\Delta$, *then the following refinement of the quadratic triangle inequality is valid*

$$\begin{aligned}(2.6)\qquad \left(\int_a^b \|f(t)\|\,dt\right)^2 &\ge \left\|\int_a^b f(t)\,dt\right\|^2 + 2\iint_\Delta k(t,s)\,dtds\\ &\left(\ge \left\|\int_a^b f(t)\,dt\right\|^2\right).\end{aligned}$$

The equality holds in (2.6) iff the case of equality holds in (2.5) for a.e. $(t,s)\in\Delta$.

The following result holds.

Theorem 3. *Let* $f\in L([a,b];H)$ *be such that there exists* $M\ge 1\ge m\ge 0$ *such that either*

$$(2.7)\qquad \operatorname{Re}\langle Mf(s) - f(t), f(t) - mf(s)\rangle \ge 0 \quad \textit{for a.e.}\quad (t,s)\in\Delta,$$

or, equivalently,

$$\left\| f(t) - \frac{M+m}{2} f(s) \right\| \leq \frac{1}{2}(M-m)\|f(s)\| \quad \text{for a.e. } (t,s) \in \Delta. \tag{2.8}$$

Then we have the inequality:

$$\left(\int_a^b \|f(t)\| \, dt \right)^2 \leq \left\| \int_a^b f(t) \, dt \right\|^2 + \frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \int_a^b (s-a) \|f(s)\|^2 \, ds. \tag{2.9}$$

The case of equality holds in (2.9) if and only if

$$\|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s) \rangle = \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|f(s)\|^2 \tag{2.10}$$

for a.e. $(t,s) \in \Delta$.

Proof. Firstly, observe that, in an inner product space $(H; \langle \cdot, \cdot \rangle)$ and for $x, z, Z \in H$, the following statements are equivalent

(i) $\operatorname{Re}\langle Z - x, x - z \rangle \geq 0$
 and
(ii) $\left\| x - \frac{Z+z}{2} \right\| \leq \frac{1}{2} \|Z - z\|$.

This shows that (2.7) and (2.8) are obviously equivalent.

Now, taking the square in (2.8), we get

$$\|f(t)\|^2 + \left(\frac{M+m}{2} \right)^2 \|f(s)\|^2 \leq 2 \operatorname{Re}\left\langle f(t), \frac{M+m}{2} f(s) \right\rangle + \frac{1}{4}(M-m)^2 \|f(s)\|^2,$$

for a.e. $(t,s) \in \Delta$, and obviously, since

$$2\left(\frac{M+m}{2} \right) \|f(t)\| \|f(s)\| \leq \|f(t)\|^2 + \left(\frac{M+m}{2} \right)^2 \|f(s)\|^2,$$

we deduce that

$$\begin{aligned} & 2\left(\frac{M+m}{2} \right) \|f(t)\| \|f(s)\| \\ \leq \; & 2 \operatorname{Re}\left\langle f(t), \frac{M+m}{2} f(s) \right\rangle + \frac{1}{4}(M-m)^2 \|f(s)\|^2, \end{aligned}$$

giving the much simpler inequality:

$$\|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s) \rangle \leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|f(s)\|^2 \tag{2.11}$$

for a.e. $(t,s) \in \Delta$.

Applying Lemma 1 for $k(t,s) := \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|f(s)\|^2$, we deduce

$$\left(\int_a^b \|f(t)\| \, dt \right)^2 \leq \left\| \int_a^b f(t) \, dt \right\|^2 + \frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \iint_\Delta \|f(s)\|^2 \, ds \tag{2.12}$$

with equality if and only if (2.11) holds for a.e. $(t,s) \in \Delta$.

Since

$$\iint_{\Delta} \|f(s)\|^2 ds = \int_a^b \left(\int_a^s \|f(s)\|^2 dt \right) ds = \int_a^b (s-a) \|f(s)\|^2 ds,$$

then by (2.12) we deduce the desired result (2.9). ∎

Another result which is similar to the one above is incorporated in the following theorem.

Theorem 4. *With the assumptions of Theorem 3, we have*

$$\left(\int_a^b \|f(t)\| dt \right)^2 - \left\| \int_a^b f(t) dt \right\|^2 \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{Mm}} \left\| \int_a^b f(t) dt \right\|^2 \tag{2.13}$$

or, equivalently,

$$\int_a^b \|f(t)\| dt \le \left(\frac{M+m}{2\sqrt{Mm}} \right)^{\frac{1}{2}} \left\| \int_a^b f(t) dt \right\|. \tag{2.14}$$

The case of equality holds in (2.13) or (2.14) if and only if

$$\|f(t)\| \|f(s)\| = \frac{M+m}{2\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle, \tag{2.15}$$

for a.e. $(t,s) \in \Delta$.

Proof. From (2.7), we deduce

$$\|f(t)\|^2 + Mm \|f(s)\|^2 \le (M+m) \operatorname{Re} \langle f(t), f(s) \rangle$$

for a.e. $(t,s) \in \Delta$. Dividing by $\sqrt{Mm} > 0$, we deduce

$$\frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm} \|f(s)\|^2 \le \frac{M+m}{\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle$$

and, obviously, since

$$2 \|f(t)\| \|f(s)\| \le \frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm} \|f(s)\|^2,$$

hence

$$\|f(t)\| \|f(s)\| \le \frac{M+m}{\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle$$

for a.e. $(t,s) \in \Delta$, giving

$$\|f(t)\| \|f(s)\| - \operatorname{Re} \langle f(t), f(s) \rangle \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle.$$

Applying Lemma 1 for $k(t,s) := \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle$, we deduce

$$\left(\int_a^b \|f(t)\| dt \right)^2 \le \left\| \int_a^b f(t) dt \right\|^2 + \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{Mm}} \operatorname{Re} \langle f(t), f(s) \rangle. \tag{2.16}$$

On the other hand, since

$$\operatorname{Re} \langle f(t), f(s) \rangle = \operatorname{Re} \langle f(s), f(t) \rangle \quad \text{for any} \quad (t,s) \in [a,b]^2,$$

hence

$$\iint_{\Delta} \operatorname{Re}\langle f(t), f(s)\rangle\, dtds = \frac{1}{2}\int_a^b\int_a^b \operatorname{Re}\langle f(t), f(s)\rangle\, dtds$$
$$= \frac{1}{2}\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \int_a^b f(s)\,ds\right\rangle$$
$$= \frac{1}{2}\left\|\int_a^b f(t)\,dt\right\|^2$$

and thus, from (2.16), we get (2.13).

The equivalence between (2.13) and (2.14) is obvious and we omit the details. ∎

3. Related Results

The following result also holds.

Theorem 5. *Let $f \in L([a,b];H)$ and $\gamma, \Gamma \in \mathbb{R}$ be such that either*

$$\operatorname{Re}\langle \Gamma f(s) - f(t), f(t) - \gamma f(s)\rangle \geq 0 \quad \text{for a.e.} \quad (t,s) \in \Delta, \tag{3.1}$$

or, equivalently,

$$\left\| f(t) - \frac{\Gamma+\gamma}{2} f(s)\right\| \leq \frac{1}{2}|\Gamma-\gamma|\,\|f(s)\| \quad \text{for a.e.} \quad (t,s)\in\Delta. \tag{3.2}$$

Then we have the inequality:

$$\int_a^b [(b-s)+\gamma\Gamma(s-a)]\,\|f(s)\|^2\,ds \leq \frac{\Gamma+\gamma}{2}\left\|\int_a^b f(s)\,ds\right\|^2. \tag{3.3}$$

The case of equality holds in (3.3) if and only if the case of equality holds in either (3.1) or (3.2) for a.e. $(t,s)\in\Delta$.

Proof. The inequality (3.1) is obviously equivalent to

$$\|f(t)\|^2 + \gamma\Gamma\|f(s)\|^2 \leq (\Gamma+\gamma)\operatorname{Re}\langle f(t), f(s)\rangle \tag{3.4}$$

for a.e. $(t,s)\in\Delta$.

Integrating (3.4) on Δ, we deduce

$$\int_a^b\left(\int_a^s \|f(t)\|^2\,dt\right)ds + \gamma\Gamma\int_a^b\left(\|f(s)\|^2\int_a^s dt\right)ds = (\Gamma+\gamma)\int_a^b\left(\int_a^s \operatorname{Re}\langle f(t), f(s)\rangle\,dt\right)ds. \tag{3.5}$$

It is easy to see, on integrating by parts, that

$$\int_a^b\left(\int_a^s \|f(t)\|^2\,dt\right)ds = s\int_a^s \|f(t)\|^2\,dt\Big|_a^b - \int_a^b s\|f(s)\|^2\,ds$$
$$= b\int_a^s \|f(s)\|^2\,ds - \int_a^b s\|f(s)\|^2\,ds$$
$$= \int_a^b (b-s)\|f(s)\|^2\,ds$$

and

$$\int_a^b \left(\|f(s)\|^2 \int_a^s dt \right) ds = \int_a^b (s-a) \|f(s)\|^2 ds.$$

Since

$$\begin{aligned}
\frac{d}{ds}\left(\left\| \int_a^b f(t)\,dt \right\|^2 \right) &= \frac{d}{ds}\left\langle \int_a^s f(t)\,dt, \int_a^s f(t)\,dt \right\rangle \\
&= \left\langle f(s), \int_a^s f(t)\,dt \right\rangle + \left\langle \int_a^s f(t)\,dt, f(s) \right\rangle \\
&= 2\operatorname{Re}\left\langle \int_a^s f(t)\,dt, f(s) \right\rangle,
\end{aligned}$$

hence

$$\begin{aligned}
\int_a^b \left(\int_a^s \operatorname{Re}\langle f(t), f(s)\rangle\,dt \right) ds &= \int_a^b \operatorname{Re}\left\langle \int_a^s f(t)\,dt, f(s) \right\rangle ds \\
&= \frac{1}{2}\int_a^b \frac{d}{ds}\left(\left\| \int_a^s f(t)\,dt \right\|^2 \right) ds \\
&= \frac{1}{2}\left\| \int_a^s f(t)\,dt \right\|^2 \Bigg|_a^b \\
&= \frac{1}{2}\left\| \int_a^b f(t)\,dt \right\|^2 .
\end{aligned}$$

Utilising (3.5), we deduce the desired inequality (3.3).

The case of equality is obvious and we omit the details. ∎

Remark 3. *Consider the function* $\varphi(s) := (b-s) + \gamma\Gamma(s-a)$, $s \in [a,b]$. *Obviously,*

$$\varphi(s) = (\Gamma\gamma - 1)s + b - \gamma\Gamma a.$$

Observe that, if $\Gamma\gamma \geq 1$, *then*

$$b - a = \varphi(a) \leq \varphi(s) \leq \varphi(b) = \gamma\Gamma(b-a), \qquad s \in [a,b]$$

and, if $\Gamma\gamma < 1$, *then*

$$\gamma\Gamma(b-a) \leq \varphi(s) \leq b - a, \qquad s \in [a,b].$$

Taking into account the above remark, we may state the following corollary.

Corollary 6. *Assume that* f, γ, Γ *are as in Theorem 5.*

a) *If* $\Gamma\gamma \geq 1$, *then we have the inequality*

$$(b-a)\int_a^b \|f(s)\|^2 ds \leq \frac{\Gamma+\gamma}{2}\left\| \int_a^b f(s)\,ds \right\|^2 .$$

b) *If* $0 < \Gamma\gamma < 1$, *then we have the inequality*

$$\gamma\Gamma(b-a)\int_a^b \|f(s)\|^2 ds \leq \frac{\Gamma+\gamma}{2}\left\| \int_a^b f(s)\,ds \right\|^2 .$$

4. Applications for Complex-Valued Functions

Let $f:[a,b]\to\mathbb{C}$ be a Lebesgue integrable function and $M\geq 1\geq m\geq 0$. The condition (2.7) from Theorem 3, which plays a fundamental role in the results obtained above, can be translated in this case as

$$\operatorname{Re}\left[\left(Mf(s)-f(t)\right)\left(\overline{f(t)}-m\overline{f(s)}\right)\right]\geq 0 \tag{4.1}$$

for a.e. $a\leq t\leq s\leq b$.

Since, obviously

$$\begin{aligned}&\operatorname{Re}\left[\left(Mf(s)-f(t)\right)\left(\overline{f(t)}-m\overline{f(s)}\right)\right]\\=\;&\left[\left(M\operatorname{Re}f(s)-\operatorname{Re}f(t)\right)\left(\operatorname{Re}f(t)-m\operatorname{Re}f(s)\right)\right]\\&+\left[\left(M\operatorname{Im}f(s)-\operatorname{Im}f(t)\right)\left(\operatorname{Im}f(t)-m\operatorname{Im}f(s)\right)\right]\end{aligned}$$

hence a sufficient condition for the inequality in (4.1) to hold is

$$m\operatorname{Re}f(s)\leq\operatorname{Re}f(t)\leq M\operatorname{Re}f(s)\ \text{ and }\ m\operatorname{Im}f(s)\leq\operatorname{Im}f(t)\leq M\operatorname{Im}f(s) \tag{4.2}$$

for a.e. $a\leq t\leq s\leq b$.

Utilising Theorems 3,4 and 5 we may state the following results incorporating quadratic reverses of the continuous triangle inequality:

Proposition 1. *With the above assumptions for f,M and m, and if (4.1) holds true, then we have the inequalities*

$$\left(\int_a^b|f(t)|\,dt\right)^2\leq\left|\int_a^b f(t)\,dt\right|^2+\frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\int_a^b(s-a)\,|f(s)|^2\,ds,$$

$$\int_a^b|f(t)|\,dt\leq\left(\frac{M+m}{2\sqrt{Mm}}\right)^{\frac{1}{2}}\left|\int_a^b f(t)\,dt\right|,$$

and

$$\int_a^b\left[(b-s)+mM(s-a)\right]|f(s)|^2\,ds\leq\frac{M+m}{2}\left|\int_a^b f(s)\,ds\right|^2.$$

Remark 4. *One may wonder if there are functions satisfying the condition (4.2) above. It suffices to find examples of real functions $\varphi:[a,b]\to\mathbb{R}$ verifying the following double inequality*

$$\gamma\varphi(s)\leq\varphi(t)\leq\Gamma\varphi(s) \tag{4.3}$$

for some given γ,Γ with $0\leq\gamma\leq 1\leq\Gamma<\infty$ for a.e. $a\leq t\leq s\leq b$.

For this purpose, consider $\psi:[a,b]\to\mathbb{R}$ a differentiable function on (a,b), continuous on $[a,b]$ and with the property that there exists $\Theta\geq 0\geq\theta$ such that

$$\theta\leq\psi'(u)\leq\Theta \text{ for any } u\in(a,b). \tag{4.4}$$

By Lagrange's mean value theorem, we have, for any $a\leq t\leq s\leq b$

$$\psi(s)-\psi(t)=\psi'(\xi)(s-t)$$

with $t\leq\xi\leq s$. Therefore, for $a\leq t\leq s\leq b$, by (4.4), we have the inequality

$$\theta(b-a)\leq\theta(s-t)\leq\psi(s)-\psi(t)\leq\Theta(s-t)\leq\Theta(b-a).$$

If we choose the function $\varphi:[a,b]\to\mathbb{R}$ given by

$$\varphi(t):=\exp\left[-\psi(t)\right],\ t\in[a,b],$$

and $\gamma := \exp[\theta(b-a)] \leq 1,\ \Gamma := \exp[\Theta(b-a)]$, *then (4.3) holds true for any* $a \leq t \leq s \leq b$.

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