

QUADRATIC REVERSES OF THE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

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ABSTRACT. Some sharp quadratic reverses for the generalised triangle inequality in inner product spaces and applications are given.

1. INTRODUCTION

In 1966, J.B. Diaz and F.T. Metcalf [1] proved the following reverse of the triangle inequality in the general settings of inner product spaces:

Theorem 1. *Let a be a unit vector in the inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field \mathbb{K} . Suppose that the vectors $x_i \in H \setminus \{0\}$, $i \in \{1, \dots, n\}$ satisfy*

$$(1.1) \quad 0 \leq r \leq \frac{\operatorname{Re} \langle x_i, a \rangle}{\|x_i\|}, \quad i \in \{1, \dots, n\}.$$

Then

$$(1.2) \quad r \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\|,$$

where equality holds if and only if

$$(1.3) \quad \sum_{i=1}^n x_i = r \left(\sum_{i=1}^n \|x_i\| \right) a.$$

For some similar results valid for semi-inner products in normed spaces, see [3] and [4].

In the same spirit, but providing a somewhat simpler sufficient condition with a clear geometrical meaning, we note the following result obtained by the author in [2]:

Theorem 2. *Let a be as above and $\rho \in (0, 1)$. If $x_i \in H$, $i \in \{1, \dots, n\}$ are such that*

$$(1.4) \quad \|x_i - a\| \leq \rho \quad \text{for each } i \in \{1, \dots, n\},$$

then we have the inequality

$$(1.5) \quad \sqrt{1 - \rho^2} \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\|,$$

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with equality if and only if

$$(1.6) \quad \sum_{i=1}^n x_i = \sqrt{1-\rho^2} \left(\sum_{i=1}^n \|x_i\| \right) a.$$

In a complementary direction providing reverses of the triangle inequality in its additive form, i.e., upper bounds for the nonnegative difference

$$\sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\|,$$

we note the following recent result obtained in [2]:

Theorem 3. *Let a be as above and $x_i \in H$, $k_i \geq 0$, $i \in \{1, \dots, n\}$ such that*

$$(1.7) \quad \|x_i\| - \operatorname{Re} \langle a, x_i \rangle \leq k_i \quad \text{for each } i \in \{1, \dots, n\},$$

then we have the inequality

$$(1.8) \quad 0 \leq \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \leq \sum_{i=1}^n k_i.$$

The equality holds in (1.8) if and only if

$$(1.9) \quad \sum_{i=1}^n \|x_i\| \geq \sum_{i=1}^n k_i$$

and

$$(1.10) \quad \sum_{i=1}^n x_i = \left(\sum_{i=1}^n \|x_i\| - \sum_{i=1}^n k_i \right) a.$$

Another similar result but with a simpler condition, is the following one [2].

Theorem 4. *Let a and x_i , $i \in \{1, \dots, n\}$ be as above. If $r_i > 0$, $i \in \{1, \dots, n\}$ are such that*

$$(1.11) \quad \|x_i - a\| \leq r_i \quad \text{for each } i \in \{1, \dots, n\},$$

then we have the inequality

$$(1.12) \quad 0 \leq \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \leq \frac{1}{2} \sum_{i=1}^n r_i^2.$$

The equality holds in (1.12) if and only if

$$(1.13) \quad \sum_{i=1}^n \|x_i\| \geq \frac{1}{2} \sum_{i=1}^n r_i^2$$

and

$$(1.14) \quad \sum_{i=1}^n x_i = \left(\sum_{i=1}^n \|x_i\| - \frac{1}{2} \sum_{i=1}^n r_i^2 \right) a.$$

For other inequalities related to the triangle inequality, see Chapter XVII of the book [5].

The main aim of the present paper is to point out some quadratic reverses for the generalised triangle inequality, namely, sharp upper bounds for the nonnegative differences

$$\left(\sum_{i=1}^n \|x_i\| \right)^2 - \left\| \sum_{i=1}^n x_i \right\|^2,$$

under various assumptions for the vectors $x_i \in H$, $i \in \{1, \dots, n\}$ involved. Some related results are established. Applications for vector-valued integrals in Hilbert spaces are also given.

2. QUADRATIC REVERSES OF THE TRIANGLE INEQUALITY

The following lemma holds:

Lemma 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} , $x_i \in H$, $i \in \{1, \dots, n\}$ and $k_{ij} > 0$ for $1 \leq i < j \leq n$ such that*

$$(2.1) \quad 0 \leq \|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle \leq k_{ij}$$

for $1 \leq i < j \leq n$. Then we have the following quadratic reverse of the triangle inequality

$$(2.2) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + 2 \sum_{1 \leq i < j \leq n} k_{ij}.$$

The case of equality holds in (2.2) if and only if it holds in (2.1) for each i, j with $1 \leq i < j \leq n$.

Proof. We observe that the following identity holds:

$$\begin{aligned} (2.3) \quad & \left(\sum_{i=1}^n \|x_i\| \right)^2 - \left\| \sum_{i=1}^n x_i \right\|^2 \\ &= \sum_{i,j=1}^n \|x_i\| \|x_j\| - \left\langle \sum_{i=1}^n x_i, \sum_{j=1}^n x_j \right\rangle \\ &= \sum_{i,j=1}^n \|x_i\| \|x_j\| - \sum_{i,j=1}^n \operatorname{Re} \langle x_i, x_j \rangle \\ &= \sum_{i,j=1}^n [\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle] \\ &= \sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle] + \sum_{1 \leq j < i \leq n} [\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle] \\ &= 2 \sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle]. \end{aligned}$$

Using the condition (2.1), we deduce that

$$\sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle] \leq \sum_{1 \leq i < j \leq n} k_{ij},$$

and by (2.3), we deduce the desired inequality (2.2).

The case of equality is obvious by the identity (2.3) and we omit the details. ■

Remark 1. From (2.2) one may deduce the coarser inequality that might be useful in some applications:

$$0 \leq \sum_{i=1}^n \|x_i\| - \left\| \sum_{i=1}^n x_i \right\| \leq \sqrt{2} \left(\sum_{1 \leq i < j \leq n} k_{ij} \right)^{\frac{1}{2}} \quad \left(\leq \sqrt{2} \sum_{1 \leq i < j \leq n} \sqrt{k_{ij}} \right).$$

Remark 2. If the condition (2.1) is replaced with the following refinement of Schwarz's inequality:

$$(2.4) \quad (0 \leq) \delta_{ij} \leq \|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle \quad \text{for } 1 \leq i < j \leq n,$$

then the following refinement of the quadratic generalised triangle inequality is valid:

$$(2.5) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \geq \left\| \sum_{i=1}^n x_i \right\|^2 + 2 \sum_{1 \leq i < j \leq n} \delta_{ij} \quad \left(\geq \left\| \sum_{i=1}^n x_i \right\|^2 \right).$$

The equality holds in the first part of (2.5) iff the case of equality holds in (2.4) for each $1 \leq i < j \leq n$.

The following result holds.

Proposition 1. Let $(H; \langle \cdot, \cdot \rangle)$ be as above, $x_i \in H$, $i \in \{1, \dots, n\}$ and $r > 0$ such that

$$(2.6) \quad \|x_i - x_j\| \leq r$$

for $1 \leq i < j \leq n$. Then

$$(2.7) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \frac{n(n-1)}{2} r^2.$$

The case of equality holds in (2.7) if and only if

$$(2.8) \quad \|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle = \frac{1}{2} r^2$$

for each i, j with $1 \leq i < j \leq n$.

Proof. The inequality (2.6) is obviously equivalent to

$$\|x_i\|^2 + \|x_j\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j \rangle + r^2$$

for $1 \leq i < j \leq n$. Since

$$2 \|x_i\| \|x_j\| \leq \|x_i\|^2 + \|x_j\|^2, \quad 1 \leq i < j \leq n;$$

hence

$$(2.9) \quad \|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle \leq \frac{1}{2} r^2$$

for any i, j with $1 \leq i < j \leq n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{2} r^2$ and taking into account that

$$\sum_{1 \leq i < j \leq n} k_{ij} = \frac{n(n-1)}{4} r^2,$$

we deduce the desired inequality (2.7). The case of equality is also obvious by the above lemma and we omit the details. ■

In the same spirit, and if some information about the forward difference $\Delta x_k := x_{k+1} - x_k$ ($1 \leq k \leq n-1$) are available, then the following simple quadratic reverse of the generalised triangle inequality may be stated.

Corollary 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$(2.10) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \frac{n(n-1)}{2} \sum_{k=1}^{n-1} \|\Delta x_k\|.$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced in general by a smaller quantity.

Proof. Let $1 \leq i < j \leq n$. Then, obviously,

$$\|x_j - x_i\| = \left\| \sum_{k=i}^{j-1} \Delta x_k \right\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq \sum_{k=1}^{n-1} \|\Delta x_k\|.$$

Applying Proposition 1 for $r := \sum_{k=1}^{n-1} \|\Delta x_k\|$, we deduce the desired result (2.10).

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (2.10) holds with a constant $c > 0$, i.e.,

$$(2.11) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + cn(n-1) \sum_{k=1}^{n-1} \|\Delta x_k\|$$

for $n \geq 2$, $x_i \in H$, $i \in \{1, \dots, n\}$.

If we choose in (2.11), $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 2c$, giving $c \geq \frac{1}{2}$. ■

The following result providing a reverse of the quadratic generalised triangle inequality in terms of the sup-norm of the forward differences also holds.

Proposition 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$(2.12) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \frac{n^2(n^2-1)}{12} \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2.$$

The constant $\frac{1}{12}$ is best possible in (2.12).

Proof. As above, we have that

$$\|x_j - x_i\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq (j-i) \max_{1 \leq k \leq n-1} \|\Delta x_k\|,$$

for $1 \leq i < j \leq n$.

Squaring the inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j \rangle + (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2$$

for any i, j with $1 \leq i < j \leq n$, and since

$$2 \|x_i\| \|x_j\| \leq \|x_j\|^2 + \|x_i\|^2,$$

hence

$$(2.13) \quad 0 \leq \|x_i\| \|x_j\| - \operatorname{Re} \langle x_i, x_j \rangle \leq \frac{1}{2} (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2$$

for any i, j with $1 \leq i < j \leq n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{2} (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2$, we can state that

$$\left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j-i)^2 \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2.$$

However,

$$\begin{aligned} \sum_{1 \leq i < j \leq n} (j-i)^2 &= \frac{1}{2} \sum_{i,j=1}^n (j-i)^2 = n \sum_{k=1}^n k^2 - \left(\sum_{k=1}^n k \right)^2 \\ &= \frac{n^2 (n^2 - 1)}{12} \end{aligned}$$

giving the desired inequality.

To prove the sharpness of the constant, assume that (2.12) holds with a constant $D > 0$, i.e.,

$$(2.14) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + D n^2 (n^2 - 1) \max_{1 \leq k \leq n-1} \|\Delta x_k\|^2$$

for $n \geq 2$, $x_i \in H$, $i \in \{1, \dots, n\}$.

If in (2.14) we choose $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 12D$ giving $D \geq \frac{1}{12}$. ■

The following result may be stated as well.

Proposition 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality:*

$$(2.15) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j-i)^{\frac{2}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $E = 1$ in front of the double sum cannot generally be replaced by a smaller constant.

Proof. Using Hölder's inequality, we have

$$\begin{aligned} \|x_j - x_i\| &\leq \sum_{k=i}^{j-1} \|\Delta x_k\| \leq (j-i)^{\frac{1}{q}} \left(\sum_{k=i}^{j-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \\ &\leq (j-i)^{\frac{1}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}}, \end{aligned}$$

for $1 \leq i < j \leq n$.

Squaring the previous inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j \rangle + (j-i)^{\frac{2}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}},$$

for $1 \leq i < j \leq n$.

Utilising the same argument from the proof of Proposition 2, we deduce the desired inequality (2.15).

Now assume that (2.15) holds with a constant $E > 0$, i.e.,

$$\left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + E \sum_{1 \leq i < j \leq n} (j-i)^{\frac{2}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{2}{p}},$$

for $n \geq 2$ and $x_i \in H$, $i \in \{1, \dots, n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

For $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $\|e\| = 1$, we get $1 \leq E$, showing the fact that the inequality (2.15) is sharp. ■

The particular case $p = q = 2$ is of interest.

Corollary 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality:*

$$(2.16) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \frac{(n^2-1)n}{6} \sum_{k=1}^{n-1} \|\Delta x_k\|^2.$$

The constant $\frac{1}{6}$ is best possible in (2.16).

Proof. For $p = q = 2$, Proposition 3 provides the inequality

$$\left(\sum_{i=1}^n \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j-i) \sum_{k=1}^{n-1} \|\Delta x_k\|^2,$$

and since

$$\begin{aligned} \sum_{1 \leq i < j \leq n} (j-i) &= 1 + (1+2) + (1+2+3) + \dots + (1+2+\dots+n-1) \\ &= \sum_{k=1}^{n-1} (1+2+\dots+k) = \sum_{k=1}^{n-1} \frac{k(k+1)}{2} \\ &= \frac{1}{2} \left[\frac{(n-1)n(2n-1)}{6} + \frac{n(n-1)}{2} \right] \\ &= \frac{n(n^2-1)}{6}, \end{aligned}$$

hence the inequality (2.15) is proved. The best constant may be shown in the same way as above but we omit the details. ■

Finally, we may state and prove the following different result.

Theorem 5. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space, $y_i \in H$, $i \in \{1, \dots, n\}$ and $M \geq m > 0$ are such that either*

$$(2.17) \quad \operatorname{Re} \langle My_j - y_i, y_i - my_j \rangle \geq 0 \quad \text{for } 1 \leq i < j \leq n,$$

or, equivalently,

$$(2.18) \quad \left\| y_i - \frac{M+m}{2} y_j \right\| \leq \frac{1}{2} (M-m) \|y_j\| \quad \text{for } 1 \leq i < j \leq n.$$

Then we have the inequality

$$(2.19) \quad \left(\sum_{i=1}^n \|y_i\| \right)^2 \leq \left\| \sum_{i=1}^n y_i \right\|^2 + \frac{1}{2} \cdot \frac{(M-m)^2}{M+m} \sum_{k=1}^{n-1} k \|y_{k+1}\|^2.$$

The case of equality holds in (2.19) if and only if

$$(2.20) \quad \|y_i\| \|y_j\| - \operatorname{Re} \langle y_i, y_j \rangle = \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|y_j\|^2$$

for each i, j with $1 \leq i < j \leq n$.

Proof. Firstly, observe that, in an inner product space $(H; \langle \cdot, \cdot \rangle)$ and for $x, z, Z \in H$, the following statements are equivalent:

- (i) $\operatorname{Re} \langle Z - x, x - z \rangle \geq 0$
- (ii) $\|x - \frac{Z+z}{2}\| \leq \frac{1}{2} \|Z - z\|$.

This shows that (2.17) and (2.18) are obviously equivalent.

Now, taking the square in (2.18), we get

$$\|y_i\|^2 + \frac{(M-m)^2}{M+m} \|y_j\|^2 \leq 2 \operatorname{Re} \left\langle y_i, \frac{M+m}{2} y_j \right\rangle + \frac{1}{n} (M-m)^2 \|y_j\|^2$$

for $1 \leq i < j \leq n$, and since, obviously,

$$2 \left(\frac{M+m}{2} \right) \|y_i\| \|y_j\| \leq \|y_i\|^2 + \frac{(M-m)^2}{M+m} \|y_j\|^2,$$

hence

$$2 \left(\frac{M+m}{2} \right) \|y_i\| \|y_j\| \leq 2 \operatorname{Re} \left\langle y_i, \frac{M+m}{2} y_j \right\rangle + \frac{1}{n} (M-m)^2 \|y_j\|^2,$$

giving the much simpler inequality

$$(2.21) \quad \|y_i\| \|y_j\| - \operatorname{Re} \langle y_i, y_j \rangle \leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|y_j\|^2,$$

for $1 \leq i < j \leq n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \|y_j\|^2$, we deduce

$$(2.22) \quad \left(\sum_{i=1}^n \|y_i\| \right)^2 \leq \left\| \sum_{i=1}^n y_i \right\|^2 + \frac{1}{2} \frac{(M-m)^2}{M+m} \sum_{1 \leq i < j \leq n} \|y_j\|^2$$

with equality if and only if (2.21) holds for each i, j with $1 \leq i < j \leq n$.

Since

$$\begin{aligned}
 \sum_{1 \leq i < j \leq n} \|y_j\|^2 &= \sum_{1 < j \leq n} \|y_j\|^2 + \sum_{2 < j \leq n} \|y_j\|^2 + \cdots + \sum_{n-1 < j \leq n} \|y_j\|^2 \\
 &= \sum_{j=2}^n \|y_j\|^2 + \sum_{j=3}^n \|y_j\|^2 + \cdots + \sum_{j=n-1}^n \|y_j\|^2 + \|y_n\|^2 \\
 &= \sum_{j=2}^n (j-1) \|y_j\|^2 = \sum_{k=1}^{n-1} k \|y_{k+1}\|^2,
 \end{aligned}$$

hence the inequality (2.19) is obtained. ■

3. FURTHER QUADRATIC REFINEMENTS OF THE TRIANGLE INEQUALITY

The following lemma is of interest in itself as well.

Lemma 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} , $x_i \in H$, $i \in \{1, \dots, n\}$ and $k \geq 1$ with the property that:*

$$(3.1) \quad \|x_i\| \|x_j\| \leq k \operatorname{Re} \langle x_i, x_j \rangle,$$

for each i, j with $1 \leq i < j \leq n$. Then

$$(3.2) \quad \left(\sum_{i=1}^n \|x_i\| \right)^2 + (k-1) \sum_{i=1}^n \|x_i\|^2 \leq k \left\| \sum_{i=1}^n x_i \right\|^2.$$

The equality holds in (3.2) if and only if it holds in (3.1) for each i, j with $1 \leq i < j \leq n$.

Proof. Firstly, let us observe that the following identity holds true:

$$\begin{aligned}
 (3.3) \quad & \left(\sum_{i=1}^n \|x_i\| \right)^2 - k \left\| \sum_{i=1}^n x_i \right\|^2 \\
 &= \sum_{i,j=1}^n \|x_i\| \|x_j\| - k \left\langle \sum_{i=1}^n x_i, \sum_{j=1}^n x_j \right\rangle \\
 &= \sum_{i,j=1}^n [\|x_i\| \|x_j\| - k \operatorname{Re} \langle x_i, x_j \rangle] \\
 &= 2 \sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - k \operatorname{Re} \langle x_i, x_j \rangle] + (1-k) \sum_{i=1}^n \|x_i\|^2,
 \end{aligned}$$

since, obviously, $\operatorname{Re} \langle x_i, x_j \rangle = \operatorname{Re} \langle x_j, x_i \rangle$ for any $i, j \in \{1, \dots, n\}$.

Using the assumption (3.1), we obtain

$$\sum_{1 \leq i < j \leq n} [\|x_i\| \|x_j\| - k \operatorname{Re} \langle x_i, x_j \rangle] \leq 0$$

and thus, from (3.3), we deduce the desired inequality (3.2).

The case of equality is obvious by the identity (3.3) and we omit the details. ■

Remark 3. The inequality (3.2) provides the following reverse of the quadratic generalised triangle inequality:

$$(3.4) \quad 0 \leq \left(\sum_{i=1}^n \|x_i\| \right)^2 - \sum_{i=1}^n \|x_i\|^2 \leq k \left[\left\| \sum_{i=1}^n x_i \right\|^2 - \sum_{i=1}^n \|x_i\|^2 \right].$$

Remark 4. Since $k = 1$ and $\sum_{i=1}^n \|x_i\|^2 \geq 0$, hence by (3.2) one may deduce the following reverse of the triangle inequality

$$(3.5) \quad \sum_{i=1}^n \|x_i\| \leq \sqrt{k} \left\| \sum_{i=1}^n x_i \right\|,$$

provided (3.1) holds true for $1 \leq i < j \leq n$.

The following corollary providing a better bound for $\sum_{i=1}^n \|x_i\|$, holds.

Corollary 3. With the assumptions in Lemma 2, one has the inequality:

$$(3.6) \quad \sum_{i=1}^n \|x_i\| \leq \sqrt{\frac{nk}{n+k-1}} \left\| \sum_{i=1}^n x_i \right\|.$$

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality

$$n \sum_{i=1}^n \|x_i\|^2 \geq \left(\sum_{i=1}^n \|x_i\| \right)^2$$

we get

$$(3.7) \quad (k-1) \sum_{i=1}^n \|x_i\|^2 + \left(\sum_{i=1}^n \|x_i\| \right)^2 \geq \left(\frac{k-1}{n} + 1 \right) \left(\sum_{i=1}^n \|x_i\| \right)^2.$$

Consequently, by (3.7) and (3.2) we deduce

$$k \left\| \sum_{i=1}^n x_i \right\|^2 \geq \frac{n+k-1}{n} \left(\sum_{i=1}^n \|x_i\| \right)^2$$

giving the desired inequality (3.6). ■

The following result may be stated as well.

Theorem 6. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H \setminus \{0\}$, $i \in \{1, \dots, n\}$, $\rho \in (0, 1)$, such that

$$(3.8) \quad \left\| x_i - \frac{x_j}{\|x_j\|} \right\| \leq \rho \quad \text{for } 1 \leq i < j \leq n.$$

Then we have the inequality

$$(3.9) \quad \sqrt{1-\rho^2} \left(\sum_{i=1}^n \|x_i\| \right)^2 + \left(1 - \sqrt{1-\rho^2} \right) \sum_{i=1}^n \|x_i\|^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2.$$

The case of equality holds in (3.9) iff

$$(3.10) \quad \|x_i\| \|x_j\| = \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re} \langle x_i, x_j \rangle$$

for any $1 \leq i < j \leq n$.

Proof. The condition (3.1) is obviously equivalent to

$$\|x_i\|^2 + 1 - \rho^2 \leq 2 \operatorname{Re} \left\langle x_i, \frac{x_j}{\|x_j\|} \right\rangle$$

for each $1 \leq i < j \leq n$.

Dividing by $\sqrt{1 - \rho^2} > 0$, we deduce

$$(3.11) \quad \frac{\|x_i\|^2}{\sqrt{1 - \rho^2}} + \sqrt{1 - \rho^2} \leq \frac{2}{\sqrt{1 - \rho^2}} \operatorname{Re} \left\langle x_i, \frac{x_j}{\|x_j\|} \right\rangle,$$

for $1 \leq i < j \leq n$.

On the other hand, by the elementary inequality

$$(3.12) \quad \frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq}, \quad p, q \geq 0, \alpha > 0$$

we have

$$(3.13) \quad 2\|x_i\| \leq \frac{\|x_i\|^2}{\sqrt{1 - \rho^2}} + \sqrt{1 - \rho^2}.$$

Making use of (3.11) and (3.13), we deduce that

$$\|x_i\| \|x_j\| \leq \frac{1}{\sqrt{1 - \rho^2}} \operatorname{Re} \langle x_i, x_j \rangle$$

for $1 \leq i < j \leq n$.

Now, applying Lemma 1 for $k = \frac{1}{\sqrt{1 - \rho^2}}$, we deduce the desired result. ■

Remark 5. If we assume that $\|x_i\| = 1$, $i \in \{1, \dots, n\}$, satisfying the simpler condition

$$(3.14) \quad \|x_j - x_i\| \leq \rho \quad \text{for } 1 \leq i < j \leq n,$$

then, from (3.9), we deduce the following lower bound for $\|\sum_{i=1}^n x_i\|$, namely

$$(3.15) \quad \left[n + n(n-1)\sqrt{1 - \rho^2} \right]^{\frac{1}{2}} \leq \left\| \sum_{i=1}^n x_i \right\|.$$

The equality holds in (3.15) iff $\sqrt{1 - \rho^2} = \operatorname{Re} \langle x_i, x_j \rangle$ for $1 \leq i < j \leq n$.

Remark 6. Under the hypothesis of Proposition 3, we have the coarser but simpler reverse of the triangle inequality

$$(3.16) \quad \sqrt[4]{1 - \rho^2} \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\|.$$

Also, applying Corollary 3 for $k = \frac{1}{\sqrt{1 - \rho^2}}$, we can state that

$$(3.17) \quad \sum_{i=1}^n \|x_i\| \leq \sqrt{\frac{n}{n\sqrt{1 - \rho^2} + 1 - \sqrt{1 - \rho^2}}} \left\| \sum_{i=1}^n x_i \right\|,$$

provided $x_i \in H$ satisfy (3.8) for $1 \leq i < j \leq n$.

In the same manner, we can state and prove the following reverse of the quadratic generalised triangle inequality.

Theorem 7. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} , $x_i \in H$, $i \in \{1, \dots, n\}$ and $M \geq m > 0$ such that either

$$(3.18) \quad \operatorname{Re} \langle Mx_j - x_i, x_i - mx_j \rangle \geq 0 \quad \text{for } 1 \leq i < j \leq n,$$

or, equivalently,

$$(3.19) \quad \left\| x_i - \frac{M+m}{2} x_j \right\| \leq \frac{1}{2} (M-m) \|x_j\| \quad \text{for } 1 \leq i < j \leq n$$

hold. Then

$$(3.20) \quad \frac{2\sqrt{mM}}{M+m} \left(\sum_{i=1}^n \|x_i\| \right)^2 + \frac{(\sqrt{M} - \sqrt{m})^2}{M+m} \sum_{i=1}^n \|x_i\|^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2.$$

The case of equality holds in (3.20) if and only if

$$(3.21) \quad \|x_i\| \|x_j\| = \frac{M+m}{2\sqrt{mM}} \operatorname{Re} \langle x_i, x_j \rangle \quad \text{for } 1 \leq i < j \leq n.$$

Proof. From (3.18), observe that

$$(3.22) \quad \|x_i\|^2 + Mm \|x_j\|^2 \leq (M+m) \operatorname{Re} \langle x_i, x_j \rangle,$$

for $1 \leq i < j \leq n$. Dividing (3.22) by $\sqrt{mM} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM} \|x_j\|^2 \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re} \langle x_i, x_j \rangle,$$

and since, obviously

$$2 \|x_i\| \|x_j\| \leq \frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM} \|x_j\|^2$$

hence

$$\|x_i\| \|x_j\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re} \langle x_i, x_j \rangle, \quad \text{for } 1 \leq i < j \leq n.$$

Applying Lemma 2 for $k = \frac{M+m}{2\sqrt{mM}} \geq 1$, we deduce the desired result. ▀

Remark 7. We also must note that a simpler but coarser inequality that can be obtained from (3.20) is

$$\left(\frac{2\sqrt{mM}}{M+m} \right)^{\frac{1}{2}} \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\|,$$

provided (3.18) holds true.

Finally, a different result related to the generalised triangle inequality is incorporated in the following theorem.

Theorem 8. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} , $\eta > 0$ and $x_i \in H$, $i \in \{1, \dots, n\}$ with the property that

$$(3.23) \quad \|x_j - x_i\| \leq \eta < \|x_j\| \quad \text{for each } i, j \in \{1, \dots, n\}.$$

Then we have the following reverse of the triangle inequality

$$(3.24) \quad \sum_{i=1}^n \sqrt{\|x_i\|^2 - \eta^2} \leq \frac{\left\| \sum_{i=1}^n x_i \right\|^2}{\sum_{i=1}^n \|x_i\|}.$$

The equality holds in (3.24) iff

$$(3.25) \quad \|x_i\| \sqrt{\|x_j\|^2 - \eta^2} = \operatorname{Re} \langle x_i, x_j \rangle \quad \text{for each } i, j \in \{1, \dots, n\}.$$

Proof. From (3.23), we have

$$\|x_i\|^2 - 2 \operatorname{Re} \langle x_i, x_j \rangle + \|x_j\|^2 \leq \eta^2,$$

giving

$$\|x_i\|^2 + \|x_j\|^2 - \eta^2 \leq 2 \operatorname{Re} \langle x_i, x_j \rangle, \quad i, j \in \{1, \dots, n\}.$$

On the other hand,

$$2 \|x_i\| \sqrt{\|x_j\|^2 - \eta^2} \leq \|x_i\|^2 + \|x_j\|^2 - \eta^2, \quad i, j \in \{1, \dots, n\}$$

and thus

$$\|x_i\| \sqrt{\|x_j\|^2 - \eta^2} \leq \operatorname{Re} \langle x_i, x_j \rangle, \quad i, j \in \{1, \dots, n\}.$$

Summing over $i, j \in \{1, \dots, n\}$, we deduce the desired inequality (3.24).

The case of equality is also obvious from the above, and we omit the details. ■

4. APPLICATIONS FOR VECTOR-VALUED INTEGRAL INEQUALITIES

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over the real or complex number field, $[a, b]$ a compact interval in \mathbb{R} and $\eta : [a, b] \rightarrow [0, \infty)$ a Lebesgue integrable function on $[a, b]$ with the property that $\int_a^b \eta(t) dt = 1$. If, by $L_\eta([a, b]; H)$ we denote the Hilbert space of all Bochner measurable functions $f : [a, b] \rightarrow H$ with the property that $\int_a^b \eta(t) \|f(t)\|^2 dt < \infty$, then the norm $\|\cdot\|_\eta$ of this space is generated by the inner product $\langle \cdot, \cdot \rangle_\eta : H \times H \rightarrow \mathbb{K}$ defined by

$$\langle f, g \rangle_\eta := \int_a^b \eta(t) \langle f(t), g(t) \rangle dt.$$

The following proposition providing a reverse of the integral generalised triangle inequality may be stated.

Proposition 4. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\eta : [a, b] \rightarrow [0, \infty)$ as above. If $g \in L_\eta([a, b]; H)$ is so that $\int_a^b \eta(t) \|g(t)\|^2 dt = 1$ and $f_i \in L_\eta([a, b]; H)$, $i \in \{1, \dots, n\}$, and $M \geq m > 0$ are so that either*

$$(4.1) \quad \operatorname{Re} \langle M f_j(t) - f_i(t), f_i(t) - m f_j(t) \rangle \geq 0$$

or, equivalently,

$$\left\| f_i(t) - \frac{m+M}{2} f_j(t) \right\| \leq \frac{1}{2} (M-m) \|f_j(t)\|$$

for a.e. $t \in [a, b]$ and $1 \leq i < j \leq n$, then we have the inequality

$$(4.2) \quad \left[\sum_{i=1}^n \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \right]^2 \\ \leq \int_a^b \eta(t) \left\| \sum_{i=1}^n f_i(t) \right\|^2 dt \\ + \frac{1}{2} \cdot \frac{(M-m)^2}{m+M} \int_a^b \eta(t) \left(\sum_{k=1}^{n-1} k \|f_{k+1}(t)\|^2 \right) dt.$$

The case of equality holds in (4.2) if and only if

$$\begin{aligned} & \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \left(\int_a^b \eta(t) \|f_j(t)\|^2 dt \right)^{1/2} \\ & - \int_a^b \eta(t) \operatorname{Re} \langle f_i(t), f_j(t) \rangle dt \\ & = \frac{1}{4} \cdot \frac{(M-m)^2}{m+M} \int_a^b \eta(t) \|f_j(t)\|^2 dt \end{aligned}$$

for each i, j with $1 \leq i < j \leq n$.

Proof. We observe that

$$\begin{aligned} & \operatorname{Re} \langle Mf_j - f_i, f_i - mf_j \rangle_\eta \\ & = \int_a^b \eta(t) \operatorname{Re} \langle Mf_j(t) - f_i(t), f_i(t) - mf_j(t) \rangle dt \geq 0 \end{aligned}$$

for any i, j with $1 \leq i < j \leq n$.

Applying Theorem 5 for the Hilbert space $L_\eta([a, b]; H)$ and for $y_i = f_i, i \in \{1, \dots, n\}$, we deduce the desired result. ■

Another integral inequality incorporated in the following proposition holds:

Proposition 5. *With the assumptions of Proposition 4, we have*

$$\begin{aligned} (4.3) \quad & \frac{2\sqrt{mM}}{m+M} \left[\sum_{i=1}^n \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \right]^2 \\ & + \frac{(\sqrt{M} - \sqrt{m})^2}{m+M} \sum_{i=1}^n \int_a^b \eta(t) \|f_i(t)\|^2 dt \\ & \leq \int_a^b \eta(t) \left\| \sum_{i=1}^n f_i(t) \right\|^2 dt. \end{aligned}$$

The case of equality holds in (4.3) if and only if

$$\begin{aligned} & \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \left(\int_a^b \eta(t) \|f_j(t)\|^2 dt \right)^{1/2} \\ & = \frac{M+m}{2\sqrt{mM}} \int_a^b \eta(t) \operatorname{Re} \langle f_i(t), f_j(t) \rangle dt \end{aligned}$$

for any i, j with $1 \leq i < j \leq n$.

The proof is obvious by Theorem 7 and we omit the details.

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