REVERSES OF THE CONTINUOUS TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN BANACH SPACES

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Abstract. Some reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in Banach spaces are given. Applications for complex-valued functions are considered as well.

1. Introduction

Let $f:[a,b]\to\mathbb{K}$, $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality, which is the continuous version of the *triangle inequality*

$$\left|\int_a^b f(x)\,dx\right|\leq\int_a^b|f(x)|\,dx, \tag{1.1}$$

plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [8, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [6]. It can be stated as

$$\cos\theta\int_a^b|f(x)|\,dx\leq\left|\int_a^b f(x)\,dx\right| \tag{1.2}$$

provided

$$-\theta\leq\arg f(x)\leq\theta,\quad x\in[a,b]$$

for given $\theta\in\left(0,\frac{\pi}{2}\right)$.

This result has recently been extended by the author for the case of Bochner integrable functions with values in a Hilbert space H. If by $L([a,b];H)$, we denote the space of Bochner integrable functions with values in a Hilbert space H, i.e., we recall that $f\in L([a,b];H)$ if and only if $f:[a,b]\to H$ is Bochner measurable on $[a,b]$ and the Lebesgue integral $\int_a^b\|f(t)\|\,dt$ is finite, then

$$\int_a^b\|f(t)\|\,dt\leq K\left\|\int_a^b f(t)\,dt\right\|, \tag{1.3}$$

provided that f satisfies the condition

$$\|f(t)\|\leq K\operatorname{Re}\langle f(t),e\rangle\quad\text{for a.e. } t\in[a,b], \tag{1.4}$$

Date: June 21, 2004.

2000 *Mathematics Subject Classification.* 46B05, 46C05, 26D15, 26D10.

Key words and phrases. Triangle inequality, Reverse inequality, Banach spaces, Hilbert spaces, Bochner integral.

where $e \in H$, $\|e\| = 1$ and $K \geq 1$ are given. The case of equality holds in (1.4) if and only if

$$\int_a^b f(t)\,dt = \frac{1}{K}\left(\int_a^b \|f(t)\|\,dt\right)e. \tag{1.5}$$

As some natural consequences of the above results, we have noted in [4] that, if $\rho \in (0,1)$ and $f \in L([a,b];H)$ are such that

$$\|f(t) - e\| \leq \rho \quad \text{for a.e. } t \in [a,b], \tag{1.6}$$

then

$$\sqrt{1-\rho^2}\int_a^b \|f(t)\|\,dt \leq \left\|\int_a^b f(t)\,dt\right\| \tag{1.7}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \sqrt{1-\rho^2}\left(\int_a^b \|f(t)\|\,dt\right)\cdot e.$$

Also, for e as above and if $M \geq m > 0$, $f \in L([a,b];H)$ such that either

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \geq 0 \tag{1.8}$$

or, equivalently,

$$\left\|f(t) - \frac{M+m}{2}e\right\| \leq \frac{1}{2}(M-m) \tag{1.9}$$

for a.e. $t \in [a,b]$, then

$$\int_a^b \|f(t)\|\,dt \leq \frac{M+m}{2\sqrt{mM}}\left\|\int_a^b f(t)\,dt\right\|, \tag{1.10}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m}\left(\int_a^b \|f(t)\|\,dt\right)\cdot e.$$

The main aim of the present paper is to extend for the case of Banach spaces the integral inequalities mentioned above. Applications for complex-valued functions are given as well.

2. Reverses of the Continuous Triangle Inequality

Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field. Then one has the following reverse of the continuous triangle inequality.

Theorem 1. *Let F be a continuous linear functional of unit norm on X. Suppose that the function $f : [a,b] \to X$ is Bochner integrable on $[a,b]$ and there exists a $r \geq 0$ such that*

$$r\|f(t)\| \leq \operatorname{Re} F(f(t)) \quad \text{for a.e. } t \in [a,b]. \tag{2.1}$$

Then

$$r\int_a^b \|f(t)\|\,dt \leq \left\|\int_a^b f(t)\,dt\right\|. \tag{2.2}$$

where equality holds in (2.2) if and only if both

$$F\left(\int_a^b f(t)\,dt\right) = r\int_a^b \|f(t)\|\,dt \tag{2.3}$$

and

$$F\left(\int_a^b f(t)\,dt\right) = \left\|\int_a^b f(t)\,dt\right\|. \tag{2.4}$$

Proof. Since the norm of F is one, then

$$|F(x)| \le \|x\| \quad \text{for any} \quad x \in X.$$

Applying this inequality for the vector $\int_a^b f(t)\,dt$, we get

$$\begin{aligned} \left\|\int_a^b f(t)\,dt\right\| &\ge \left|F\left(\int_a^b f(t)\,dt\right)\right| \\ &\ge \left|\operatorname{Re} F\left(\int_a^b f(t)\,dt\right)\right| = \left|\int_a^b \operatorname{Re} F(f(t))\,dt\right|. \end{aligned} \tag{2.5}$$

Now, by integration of (2.1), we obtain

$$\int_a^b \operatorname{Re} F(f(t))\,dt \ge r\int_a^b \|f(t)\|\,dt, \tag{2.6}$$

and by (2.5) and (2.6) we deduce the desired inequality (2.1).

Obviously, if (2.3) and (2.4) hold true, then the equality case holds in (2.2).

Conversely, if the case of equality holds in (2.2), then it must hold in all the inequalities used before in proving this inequality. Therefore, we must have

$$r\|f(t)\| = \operatorname{Re} F(f(t)) \quad \text{for a.e. } t \in [a,b], \tag{2.7}$$

$$\operatorname{Im} F\left(\int_a^b f(t)\,dt\right) = 0 \tag{2.8}$$

and

$$\left\|\int_a^b f(t)\,dt\right\| = \operatorname{Re} F\left(\int_a^b f(t)\,dt\right). \tag{2.9}$$

Integrating (2.7) on $[a,b]$, we get

$$r\int_a^b \|f(t)\|\,dt = \operatorname{Re} F\left(\int_a^b f(t)\,dt\right). \tag{2.10}$$

On utilising (2.10) and (2.8), we deduce (2.3) while (2.9) and (2.10) would imply (2.4), and the theorem is proved. ∎

In 1961, G. Lumer [7] introduced the following concept.

Definition 1. *Let X be a linear space over the real or complex number field $\mathbb{K}$. The mapping $[\cdot,\cdot] : X \times X \to \mathbb{K}$ is called a semi-inner product on X, if the following properties are satisfied (see also* [2, p. 17]*):*

(*i*) $[x+y,z] = [x,z] + [y,z]$ *for all* $x,y,z \in X$;

(ii) $[\lambda x, y] = \lambda [x, y]$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$;
(iii) $[x, x] \geq 0$ *for all* $x \in X$ *and* $[x, x] = 0$ *implies* $x = 0$;
(iv) $|[x, y]|^2 \leq [x, x][y, y]$ *for all* $x, y \in X$;
(v) $[x, \lambda y] = \bar{\lambda} [x, y]$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$.

It is well known that the mapping $X \ni x \longmapsto [x, x]^{\frac{1}{2}} \in \mathbb{R}$ is a norm on X and for any $y \in X$, the functional $X \ni x \overset{\varphi_y}{\longmapsto} [x, x]^{\frac{1}{2}} \in \mathbb{K}$ is a continuous linear functional on X endowed with the norm $\|\cdot\|$ generated by $[\cdot, \cdot]$. Moreover, one has $\|\varphi_y\| = \|y\|$ (see for instance [2, p. 17]).

Let $(X, \|\cdot\|)$ be a real or complex normed space. If $J : X \to_2 X^*$ is the *normalised duality mapping* defined on X, i.e., we recall that (see for instance [2, p. 1])

$$J(x) = \{\varphi \in X^* | \varphi(x) = \|\varphi\| \|x\|, \ \|\varphi\| = \|x\|\}, \quad x \in X,$$

then we may state the following representation result (see for instance [2, p. 18]):

Each semi-inner product $[\cdot, \cdot] : X \times X \to \mathbb{K}$ that generates the norm $\|\cdot\|$ of the normed linear space $(X, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$, is of the form

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \text{ for any } x, y \in X,$$

where $\tilde{J}$ is a selection of the normalised duality mapping and $\langle \varphi, x \rangle := \varphi(x)$ for $\varphi \in X^*$ and $x \in X$.

Corollary 1. *Let* $(X, \|\cdot\|)$ *be a Banach space,* $[\cdot, \cdot] : X \times X \to \mathbb{R}$ *a semi-inner product generating the norm* $\|\cdot\|$ *and* $e \in X$, $\|e\| = 1$. *Suppose that the function* $f : [a, b] \to X$ *is Bochner integrable on* $[a, b]$ *and there exists a* $r \geq 0$ *such that*

$$r \|f(t)\| \leq \operatorname{Re}[f(t), e] \quad \text{for a.e. } t \in [a, b]. \tag{2.11}$$

Then

$$r \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\| \tag{2.12}$$

where equality holds in (2.12) if and only if both

$$\left[\int_a^b f(t) \, dt, e \right] = r \int_a^b \|f(t)\| \, dt \tag{2.13}$$

and

$$\left[\int_a^b f(t) \, dt, e \right] = \left\| \int_a^b f(t) \, dt \right\|. \tag{2.14}$$

The proof follows from Theorem 1 for the continuous linear functional $F(x) = [x, e]$, $x \in X$, and we omit the details.

Before we provide a simple necessary and sufficient condition of equality in (2.12), we need to recall the concept of strictly convex normed spaces and a classical characterisation of these spaces.

Definition 2. *A normed linear space* $(X, \|\cdot\|)$ *is said to be strictly convex if for every* x, y *from* X *with* $x \neq y$ *and* $\|x\| = \|y\| = 1$, *we have* $\|\lambda x + (1 - \lambda) y\| < 1$ *for all* $\lambda \in (0, 1)$.

The following characterisation of strictly convex spaces is useful in what follows (see [1], [5], [9] or [2, p. 21]).

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$ and $[\cdot,\cdot]$ a semi-inner product generating its norm. The following statements are equivalent:*

(*i*) $(X, \|\cdot\|)$ *is strictly convex;*
(*ii*) *For every $x, y \in X$, $x, y \neq 0$ with $[x, y] = \|x\| \|y\|$, there exists a $\lambda > 0$ such that $x = \lambda y$.*

The following corollary of Theorem 1 may be stated.

Corollary 2. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space, $[\cdot,\cdot] : X \times X \to \mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e \in X$, $\|e\| = 1$. If $f : [a, b] \to X$ is Bochner integrable on $[a, b]$ and there exists a $r \geq 0$ such that (2.11) holds true, then (2.12) is valid. The case of equality holds in (2.12) if and only if*

$$\int_a^b f(t)\,dt = r\left(\int_a^b \|f(t)\|\,dt\right)e. \tag{2.15}$$

Proof. If (2.15) holds true, then, obviously

$$\left\|\int_a^b f(t)\,dt\right\| = r\left(\int_a^b \|f(t)\|\,dt\right)\|e\| = r\int_a^b \|f(t)\|\,dt,$$

which is the equality case in (2.12).

Conversely, if the equality holds in (2.12), then, by Corollary 1, we must have (2.13) and (2.14). Utilising Theorem 2, by (2.14) we can conclude that there exists a $\mu > 0$ such that

$$\int_a^b f(t)\,dt = \mu e. \tag{2.16}$$

Replacing this in (2.13), we get

$$\mu\|e\|^2 = r\int_a^b \|f(t)\|\,dt,$$

giving

$$\mu = r\int_a^b \|f(t)\|\,dt. \tag{2.17}$$

Utilising (2.16) and (2.17) we deduce (2.15) and the proof is completed. ∎

3. Reverses for m Functionals

The following result may be stated:

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. If $f : [a, b] \to X$ is a Bochner integrable function on $[a, b]$ and there exists $r_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k\|f(t)\| \leq \operatorname{Re} F_k(f(t)) \tag{3.1}$$

for each $k \in \{1, \dots, m\}$ and a.e. $t \in [a, b]$, then

$$\int_a^b \|f(t)\|\,dt \leq \frac{\left\|\sum_{k=1}^m F_k\right\|}{\sum_{k=1}^m r_k}\left\|\int_a^b f(t)\,dt\right\|. \tag{3.2}$$

The case of equality holds in (3.2) if both

$$\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left(\sum_{k=1}^{m} r_k\right)\int_a^b \|f(t)\|\,dt \tag{3.3}$$

and

$$\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\int_a^b f(t)\,dt\right\|. \tag{3.4}$$

Proof. Utilising the hypothesis (3.1), we have

$$\begin{aligned} I &:= \left|\sum_{k=1}^{m} F_k\left(\int_a^b f(t)\,dt\right)\right| \geq \left|\operatorname{Re}\left[\sum_{k=1}^{m} F_k\left(\int_a^b f(t)\,dt\right)\right]\right| \\ &\geq \operatorname{Re}\left[\sum_{k=1}^{m} F_k\left(\int_a^b f(t)\,dt\right)\right] = \sum_{k=1}^{m}\left(\int_a^b \operatorname{Re} F_k f(t)\,dt\right) \\ &\geq \left(\sum_{k=1}^{m} r_k\right)\cdot\int_a^b \|f(t)\|\,dt. \end{aligned} \tag{3.5}$$

On the other hand, by the continuity property of F_k, $k \in \{1,\dots,m\}$, we obviously have

$$I = \left|\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right)\right| \leq \left\|\sum_{k=1}^{m} F_k\right\|\left\|\int_a^b f(t)\,dt\right\|. \tag{3.6}$$

Making use of (3.5) and (3.6), we deduce (3.2).

Now, obviously, if (3.3) and (3.4) are valid, then the case of equality holds true in (3.2).

Conversely, if the equality holds in the inequality (3.2), then it must hold in all the inequalities used to prove (3.2), therefore we have

$$r_k \|f(t)\| = \operatorname{Re} F_k(f(t)) \quad \text{for each } k \in \{1,\dots,m\} \quad \text{and a.e. } t \in [a,b], \tag{3.7}$$

$$\operatorname{Im}\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = 0, \tag{3.8}$$

$$\operatorname{Re}\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\int_a^b f(t)\,dt\right\|. \tag{3.9}$$

Note that, by (3.7), on integrating on $[a,b]$ and summing over $k \in \{1,\dots,m\}$, we get

$$\operatorname{Re}\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left(\sum_{k=1}^{m} r_k\right)\int_a^b \|f(t)\|\,dt. \tag{3.10}$$

Now, (3.8) and (3.10) imply (3.3) while (3.8) and (3.9) imply (3.4), therefore the theorem is proved. ∎

The case of Hilbert spaces which provides a simpler condition for equality is of interest for applications.

Theorem 4. *Let $(X, \|\cdot\|)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k \in H \backslash \{0\}$, $k \in \{1, \ldots, m\}$. If $f : [a,b] \to H$ is a Bochner integrable function and $r_k \geq 0$, $k \in \{1, \ldots, m\}$ and $\sum_{k=1}^{m} r_k > 0$ satisfy*

$$r_k \|f(t)\| \leq \operatorname{Re} \langle f(t), e_k \rangle \tag{3.11}$$

for each $k \in \{1, \ldots, m\}$ and for a.e. $t \in [a,b]$, then

$$\int_a^b \|f(t)\| \, dt \leq \frac{\left\|\sum_{k=1}^{m} e_k\right\|}{\sum_{k=1}^{m} r_k} \left\| \int_a^b f(t) \, dt \right\|. \tag{3.12}$$

The case of equality holds in (3.12) for $f \neq 0$ a.e. on $[a,b]$ if and only if

$$\int_a^b f(t) \, dt = \frac{\left(\sum_{k=1}^{m} r_k\right) \int_a^b \|f(t)\| \, dt}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \sum_{k=1}^{m} e_k. \tag{3.13}$$

Proof. Utilising the hypothesis (3.11) and the modulus properties, we have

$$\begin{aligned} \left| \left\langle \int_a^b f(t) \, dt, \sum_{k=1}^{m} e_k \right\rangle \right| &\geq \left| \sum_{k=1}^{m} \operatorname{Re} \left\langle \int_a^b f(t) \, dt, e_k \right\rangle \right| \\ &\geq \sum_{k=1}^{m} \operatorname{Re} \left\langle \int_a^b f(t) \, dt, e_k \right\rangle \\ &= \sum_{k=1}^{m} \int_a^b \operatorname{Re} \langle f(t), e_k \rangle \, dt \\ &\geq \left(\sum_{k=1}^{m} r_k \right) \int_a^b \|f(t)\| \, dt. \end{aligned} \tag{3.14}$$

By Schwarz's inequality in Hilbert spaces applied for $\int_a^b f(t) \, dt$ and $\sum_{k=1}^{m} e_k$, we have

$$\left\| \int_a^b f(t) \, dt \right\| \left\| \sum_{k=1}^{m} e_k \right\| \geq \left| \left\langle \int_a^b f(t) \, dt, \sum_{k=1}^{m} e_k \right\rangle \right|. \tag{3.15}$$

Making use of (3.14) and (3.15), we deduce (3.12).

Now, if $f \neq 0$ a.e. on $[a,b]$, then $\int_a^b \|f(t)\| \, dt \neq 0$ and by (3.14) $\sum_{k=1}^{m} e_k \neq 0$. Obviously, if (3.13) is valid, then taking the norm we have

$$\begin{aligned} \left\| \int_a^b f(t) \, dt \right\| &= \frac{\left(\sum_{k=1}^{m} r_k\right) \int_a^b \|f(t)\| \, dt}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \left\| \sum_{k=1}^{m} e_k \right\| \\ &= \frac{\sum_{k=1}^{m} r_k}{\left\|\sum_{k=1}^{m} e_k\right\|} \int_a^b \|f(t)\| \, dt, \end{aligned}$$

i.e., the case of equality holds true in (3.12).

Conversely, if the equality case holds true in (3.12), then it must hold in all the inequalities used to prove (3.12), therefore we have

$$\operatorname{Re} \langle f(t), e_k \rangle = r_k \|f(t)\| \quad \text{for each } k \in \{1, \ldots, m\} \quad \text{and a.e. } t \in [a,b], \tag{3.16}$$

$$\left\| \int_a^b f(t) \, dt \right\| \left\| \sum_{k=1}^{m} e_k \right\| = \left| \left\langle \int_a^b f(t) \, dt, \sum_{k=1}^{m} e_k \right\rangle \right|, \tag{3.17}$$

and

$$\operatorname{Im}\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = 0. \tag{3.18}$$

From (3.16) on integrating on $[a,b]$ and summing over k from 1 to m, we get

$$\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = \left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt, \tag{3.19}$$

and then, by (3.18) and (3.19), we have

$$\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = \left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt. \tag{3.20}$$

On the other hand, by the use of the following identity in Hilbert spaces:

$$\left\| u - \frac{\langle u,v\rangle v}{\|v\|^2} \right\|^2 = \frac{\|u\|^2\|v\|^2 - |\langle u,v\rangle|^2}{\|v\|^2}, \quad v \neq 0,$$

the relation (3.17) holds true if and only if

$$\int_a^b f(t)\,dt = \frac{\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle}{\left\|\sum_{k=1}^m e_k\right\|}\sum_{k=1}^m e_k. \tag{3.21}$$

Finally, by (3.20) and (3.21) we deduce that (3.13) is also necessary for the equality case in (3.12) and the theorem is proved. ∎

Remark 1. *If $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthogonal, then (3.12) can be replaced by*

$$\int_a^b \|f(t)\|\,dt \le \frac{\left(\sum_{k=1}^m \|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m r_k}\left\|\int_a^b f(t)\,dt\right\|, \tag{3.22}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{\left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt}{\sum_{k=1}^m \|e_k\|^2}\sum_{k=1}^m e_k. \tag{3.23}$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthonormal, then (3.22) becomes

$$\int_a^b \|f(t)\|\,dt \le \frac{\sqrt{m}}{\sum_{k=1}^m r_k}\left\|\int_a^b f(t)\,dt\right\|, \tag{3.24}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{1}{m}\left(\sum_{k=1}^m r_k\right)\left(\int_a^b \|f(t)\|\,dt\right)\sum_{k=1}^m e_k. \tag{3.25}$$

The following result is of interest in itself as well.

Lemma 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$ such that*

$$\|x - a\| \le r < \|a\|. \tag{3.26}$$

Then we have the inequality:

$$\|x\| \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\langle x, a\rangle, \tag{3.27}$$

or, equivalently

$$\|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 \leq r^2 \|x\|^2. \tag{3.28}$$

The equality holds in (3.27) (or equivalently, in (3.28)) if and only if

$$\|x - a\| = r \quad and \quad \|x\|^2 + r^2 = \|a\|^2. \tag{3.29}$$

Proof. From the first part of (3.26), we have

$$\|x\|^2 + \|a\|^2 - r^2 \leq 2 \operatorname{Re}\langle x, a\rangle. \tag{3.30}$$

By the second part of (3.26) we have $\left(\|a\|^2 - r^2\right)^{\frac{1}{2}} > 0$, therefore, by (3.30), we may state that

$$0 < \frac{\|x\|^2}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \frac{2 \operatorname{Re}\langle x, a\rangle}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}}. \tag{3.31}$$

Utilising the elementary inequality

$$\frac{1}{\alpha} q + \alpha p \geq 2\sqrt{pq}, \qquad \alpha > 0,\ p > 0,\ q \geq 0;$$

with equality if and only if $\alpha = \sqrt{\frac{q}{p}}$, we may state (for $\alpha = \left(\|a\|^2 - r^2\right)^{1/2}$, $p = 1$, $q = \|x\|^2$) that

$$2\|x\| \leq \frac{\|x\|^2}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2\right)^{\frac{1}{2}}. \tag{3.32}$$

The inequality (3.27) follows now by (3.31) and (3.32).

From the above argument it is clear that the equality holds in (3.27) if and only if it holds in (3.30) and (3.32). However, the equality holds in (3.30) if and only if $\|x - a\| = r$ and in (3.32) if and only if $\left(\|a\|^2 - r^2\right)^{\frac{1}{2}} = \|x\|$. The proof is completed. ∎

Remark 2. *For $r > 0$ and $a \in H$ with $\|a\| > r$, define*

$$\mathcal{P}_r(a) := \left\{x \in H \mid \|x - a\| = r,\ \|x\|^2 + r^2 = \|a\|^2\right\}.$$

It is clear that, for a given a, the case of equality holds in (3.27) if and only if $x \in \mathcal{P}_r(a)$.

If we define

$$\mathcal{U}_r(a) := \{x \in H | x = a - re,\ \|e\| = 1,\ \langle e, a\rangle = r,\ e \in H\}$$

then we observe that $\|x - a\| = r$ and

$$\begin{aligned}\|x\|^2 &= \|a - re\|^2 = \|a\|^2 - 2r \operatorname{Re}\langle a, e\rangle + r^2 \\ &= \|a\|^2 - 2r^2 + r^2 = \|a\|^2 - r^2,\end{aligned}$$

giving that

$$\mathcal{U}_r(a) \subseteq \mathcal{P}_r(a).$$

Remark 3. *The inequality (3.28) has been established in* [3]*, but without the case of equality. In* [3]*, only the sharpness of the constant was considered.*

The following corollary of Theorem 4 may be stated as well.

Corollary 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k \in H\backslash\{0\}$, $k \in \{1,\dots,m\}$. If $f:[a,b]\to H$ is a Bochner integrable function on $[a,b]$ and $\rho_k > 0$, $k\in\{1,\dots,m\}$ with*

$$\|f(t)-e_k\| \leq \rho_k < \|e_k\| \quad \text{for each } k\in\{1,\dots,m\} \quad \text{and a.e. } t\in[a,b], \tag{3.33}$$

then

$$\int_a^b \|f(t)\|\,dt \leq \frac{\left\|\sum_{k=1}^m e_k\right\|}{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\int_a^b f(t)\,dt\right\|. \tag{3.34}$$

The case of equality holds in (3.34) if and only if

$$\int_a^b f(t)\,dt = \frac{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\left\|\sum_{k=1}^m e_k\right\|^2}\left(\int_a^b\|f(t)\|\,dt\right)\sum_{k=1}^m e_k. \tag{3.35}$$

Proof. Utilising Lemma 1, we have from (3.2) that

$$\|f(t)\|\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\langle f(t),e_k\rangle$$

for any $k\in\{1,\dots,m\}$ and a.e. $t\in[a,b]$.

Applying Theorem 4 for

$$r_k := \left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}, \quad k\in\{1,\dots,m\},$$

we deduce the desired result. ∎

Remark 4. *If $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthogonal, then (3.34) becomes*

$$\int_a^b \|f(t)\|\,dt \leq \frac{\left(\sum_{k=1}^m\|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\int_a^b f(t)\,dt\right\|, \tag{3.36}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m\|e_k\|^2}\left(\int_a^b\|f(t)\|\,dt\right)\sum_{k=1}^m e_k. \tag{3.37}$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ is assumed to be orthonormal and

$$\|f(t)-e_k\| \leq \rho_k \quad \text{for a.e. } t\in[a,b],$$

where $\rho_k\in[0,1)$, $k\in\{1,\dots,m\}$, then

$$\int_a^b \|f(t)\|\,dt \leq \frac{\sqrt{m}}{\sum_{k=1}^m\left(1-\rho_k^2\right)^{\frac{1}{2}}}\left\|\int_a^b f(t)\,dt\right\|, \tag{3.38}$$

with equality iff

$$\int_a^b f(t)\,dt = \frac{\sum_{k=1}^m \left(1-\rho_k^2\right)^{\frac{1}{2}}}{m}\left(\int_a^b \|f(t)\|\,dt\right)\sum_{k=1}^m e_k. \tag{3.39}$$

The following lemma may be stated as well.

Lemma 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x,y\in H$ and $M\geq m>0$. If*

$$\operatorname{Re}\langle My-x, x-my\rangle \geq 0 \tag{3.40}$$

or, equivalently,

$$\left\|x-\frac{m+M}{2}y\right\| \leq \frac{1}{2}(M-m)\|y\|, \tag{3.41}$$

then

$$\|x\|\,\|y\| \leq \frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x,y\rangle. \tag{3.42}$$

The equality holds in (3.42) if and only if the case of equality in (3.36) holds and

$$\|x\| = \sqrt{mM}\,\|y\|. \tag{3.43}$$

Proof. Obviously,

$$\operatorname{Re}\langle My-x, x-my\rangle = (M+m)\operatorname{Re}\langle x,y\rangle - \|x\|^2 - mM\|y\|^2.$$

Then (3.40) is clearly equivalent to

$$\frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\,\|y\|^2 \leq \frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x,y\rangle. \tag{3.44}$$

Since, obviously

$$2\|x\|\,\|y\| \leq \frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\,\|y\|^2, \tag{3.45}$$

with equality if and only if $\|x\| = \sqrt{mM}\,\|y\|$, hence (3.44) and (3.45) imply the desired inequality (3.42).

The equality holds in (3.42) if and only if it holds in both (3.44) and (3.45) and the conclusion of the lemma is obtained. ∎

Remark 5. *The inequality (3.42) has been obtained in* [3] *as a particular case of a more general result holding for some complex numbers Γ,γ instead of M and m. The case of equality was not considered in* [3], *however the constant $\frac{1}{2}$ has been shown to be the best possible.*

Finally, we may state the following corollary of Theorem 4.

Corollary 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k\in H\backslash\{0\}$, $k\in\{1,\ldots,m\}$. If $f:[a,b]\to H$ is a Bochner integrable function on $[a,b]$ and $M_k\geq\mu_k>0$, $k\in\{1,\ldots,m\}$ are such that either*

$$\operatorname{Re}\langle M_k e_k - f(t), f(t)-\mu_k e_k\rangle \geq 0 \tag{3.46}$$

or, equivalently

$$\left\|f(t)-\frac{M_k+\mu_k}{2}e_k\right\| \leq \frac{1}{2}(M_k-\mu_k)\|e_k\| \tag{3.47}$$

for each $k \in \{1, \dots, m\}$ *and a.e.* $t \in [a, b]$*, then*

$$\int_a^b \|f(t)\| \, dt \le \frac{\left\|\sum_{k=1}^m e_k\right\|}{\sum_{k=1}^m \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\|} \left\|\int_a^b f(t) \, dt\right\|. \tag{3.48}$$

The case of equality holds if and only if

$$\int_a^b f(t) \, dt = \frac{\sum_{k=1}^m \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\|}{\left\|\sum_{k=1}^m e_k\right\|^2} \left(\int_a^b \|f(t)\| \, dt\right) \cdot \sum_{k=1}^m e_k.$$

Proof. Utilising Lemma 3.9, by (3.46) we deduce

$$\|f(t)\| \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\| \le \operatorname{Re}\langle f(t), e_k\rangle$$

for each $k \in \{1, \dots, m\}$ and a.e. $t \in [a, b]$.

Applying Theorem 4 for

$$r_k := \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\|, \quad k \in \{1, \dots, m\}$$

we deduce the desired result. ∎

4. Applications for Complex-Valued Functions

Let $\mathbb{C}$ be the field of complex numbers. If $z = \operatorname{Re} z + i \operatorname{Im} z$, then by $|\cdot|_p : \mathbb{C} \to [0, \infty)$, $p \in [1, \infty]$ we define the $p-$*modulus* of z as

$$|z|_p := \begin{cases} \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} & \text{if } p = \infty, \\ \\ (|\operatorname{Re} z|^p + |\operatorname{Im} z|^p)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \end{cases}$$

where $|a|$, $a \in \mathbb{R}$ is the usual modulus of the real number a.

For $p = 2$, we recapture the usual modulus of a complex number, i.e.,

$$|z|_2 = \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2} = |z|, \quad z \in \mathbb{C}.$$

It is well known that $\left(\mathbb{C}, |\cdot|_p\right)$, $p \in [1, \infty]$ is a Banach space over the complex number field $\mathbb{C}$.

We now give some examples of inequalities for complex-valued functions that are Lebesgue integrable on using the general result obtained in Section 2.

Consider the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F : \mathbb{C} \to \mathbb{C}$, $F(z) = ez$ with $e = \alpha + i\beta$ and $|e|^2 = \alpha^2 + \beta^2 = 1$, then F is linear on $\mathbb{C}$. For $z \ne 0$, we have

$$\frac{|F(z)|}{|z|_1} = \frac{|e||z|}{|z|_1} = \frac{\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{|\operatorname{Re} z| + |\operatorname{Im} z|} \le 1.$$

Since, for $z_0 = 1$, we have $|F(z_0)| = 1$ and $|z_0|_1 = 1$, hence

$$\|F\|_1 := \sup_{z \ne 0} \frac{|F(z)|}{|z|_1} = 1,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_1)$.

Therefore we can apply Theorem 1 to state the following result for complex-valued functions.

Proposition 1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$, $f : [a, b] \to \mathbb{C}$ a Lebesgue integrable function on $[a, b]$ and $r \geq 0$ such that*

$$r\left[\left|\operatorname{Re} f(t)\right| + \left|\operatorname{Im} f(t)\right|\right] \leq \alpha \operatorname{Re} f(t) - \beta \operatorname{Im} f(t) \tag{4.1}$$

for a.e. $t \in [a, b]$. Then

$$r\left[\int_a^b \left|\operatorname{Re} f(t)\right| dt + \int_a^b \left|\operatorname{Im} f(t)\right| dt\right] \leq \left|\int_a^b \operatorname{Re} f(t)\, dt\right| + \left|\int_a^b \operatorname{Im} f(t)\, dt\right|. \tag{4.2}$$

The equality holds in (4.2) if and only if both

$$\alpha \int_a^b \operatorname{Re} f(t)\, dt - \beta \int_a^b \operatorname{Im} f(t)\, dt = r\left[\int_a^b \left|\operatorname{Re} f(t)\right| dt + \int_a^b \left|\operatorname{Im} f(t)\right| dt\right]$$

and

$$\alpha \int_a^b \operatorname{Re} f(t)\, dt - \beta \int_a^b \operatorname{Im} f(t)\, dt = \left|\int_a^b \operatorname{Re} f(t)\, dt\right| + \left|\int_a^b \operatorname{Im} f(t)\, dt\right|.$$

Now, consider the Banach space $\left(\mathbb{C}, |\cdot|_\infty\right)$. If $F(z) = dz$ with $d = \gamma + i\delta$ and $|d| = \frac{\sqrt{2}}{2}$, i.e., $\gamma^2 + \delta^2 = \frac{1}{2}$, then F is linear on $\mathbb{C}$. For $z \neq 0$ we have

$$\frac{|F(z)|}{|z|_\infty} = \frac{|d|\,|z|}{|z|_\infty} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\max\left\{|\operatorname{Re} z|, |\operatorname{Im} z|\right\}} \leq 1.$$

Since, for $z_0 = 1 + i$, we have $|F(z_0)| = 1$, $|z_0|_\infty = 1$, hence

$$\|F\|_\infty := \sup_{z \neq 0} \frac{|F(z)|}{|z|_\infty} = 1,$$

showing that F is a bounded linear functional of unit norm on $\left(\mathbb{C}, |\cdot|_\infty\right)$.

Therefore, we can apply Theorem 1, to state the following result for complex-valued functions.

Proposition 2. *Let $\gamma, \delta \in \mathbb{R}$ with $\gamma^2 + \delta^2 = \frac{1}{2}$, $f : [a, b] \to \mathbb{C}$ a Lebesgue integrable function on $[a, b]$ and $r \geq 0$ such that*

$$r \max\left\{\left|\operatorname{Re} f(t)\right|, \left|\operatorname{Im} f(t)\right|\right\} \leq \gamma \operatorname{Re} f(t) - \delta \operatorname{Im} f(t)$$

for a.e. $t \in [a, b]$. Then

$$r\int_a^b \max\left\{\left|\operatorname{Re} f(t)\right|, \left|\operatorname{Im} f(t)\right|\right\} dt \leq \max\left\{\left|\int_a^b \operatorname{Re} f(t)\, dt\right|, \left|\int_a^b \operatorname{Im} f(t)\, dt\right|\right\}. \tag{4.3}$$

The equality holds in (4.3) if and only if both

$$\gamma \int_a^b \operatorname{Re} f(t)\, dt - \delta \int_a^b \operatorname{Im} f(t)\, dt = r\int_a^b \max\left\{\left|\operatorname{Re} f(t)\right|, \left|\operatorname{Im} f(t)\right|\right\} dt$$

and

$$\gamma \int_a^b \operatorname{Re} f(t)\, dt - \delta \int_a^b \operatorname{Im} f(t)\, dt = \max\left\{\left|\int_a^b \operatorname{Re} f(t)\, dt\right|, \left|\int_a^b \operatorname{Im} f(t)\, dt\right|\right\}.$$

Now, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$. Let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = cz$ with $|c| = 2^{\frac{1}{2p} - \frac{1}{2}}$ $(p \geq 1)$. Obviously, F is linear and by Hölder's inequality

$$\frac{|F(z)|}{|z|_{2p}} = \frac{2^{\frac{1}{2p} - \frac{1}{2}} \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\left(|\operatorname{Re} z|^{2p} + |\operatorname{Im} z|^{2p}\right)^{\frac{1}{2p}}} \leq 1.$$

Since, for $z_0 = 1 + i$ we have $|F(z_0)| = 2^{\frac{1}{p}}$, $|z_0|_{2p} = 2^{\frac{1}{2p}}$ $(p \geq 1)$, hence

$$\|F\|_{2p} := \sup_{z \neq 0} \frac{|F(z)|}{|z|_{2p}} = 1,$$

showing that F is a bounded linear functional of unit norm on $\left(\mathbb{C}, |\cdot|_{2p}\right)$, $(p \geq 1)$. Therefore on using Theorem 1, we may state the following result.

Proposition 3. *Let $\varphi, \phi \in \mathbb{R}$ with $\varphi^2 + \phi^2 = 2^{\frac{1}{2p} - \frac{1}{2}}$ $(p \geq 1)$, $f : [a, b] \to \mathbb{C}$ a Lebesgue integrable function on $[a, b]$ and $r \geq 0$ such that*

$$r \left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}} \leq \varphi \operatorname{Re} f(t) - \phi \operatorname{Im} f(t)$$

for a.e. $t \in [a, b]$, then

$$r \int_a^b \left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}} dt \leq \left[\left|\int_a^b \operatorname{Re} f(t)\, dt\right|^{2p} + \left|\int_a^b \operatorname{Im} f(t)\, dt\right|^{2p}\right]^{\frac{1}{2p}}, \quad (p \geq 1) \tag{4.4}$$

where equality holds in (4.4) if and only if both

$$\varphi \int_a^b \operatorname{Re} f(t)\, dt - \phi \int_a^b \operatorname{Im} f(t)\, dt = r \int_a^b \left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}} dt$$

and

$$\varphi \int_a^b \operatorname{Re} f(t)\, dt - \phi \int_a^b \operatorname{Im} f(t)\, dt = \left[\left|\int_a^b \operatorname{Re} f(t)\, dt\right|^{2p} + \left|\int_a^b \operatorname{Im} f(t)\, dt\right|^{2p}\right]^{\frac{1}{2p}}.$$

Remark 6. *If $p = 1$ above, and*

$$r|f(t)| \leq \varphi \operatorname{Re} f(t) - \psi \operatorname{Im} f(t) \quad \text{for a.e. } t \in [a, b],$$

provided φ, $\psi \in \mathbb{R}$ and $\varphi^2 + \psi^2 = 1, r \geq 0$, then we have a reverse of the classical continuous triangle inequality for modulus:

$$r \int_a^b |f(t)|\, dt \leq \left|\int_a^b f(t)\, dt\right|,$$

with equality iff

$$\varphi \int_a^b \operatorname{Re} f(t)\, dt - \psi \int_a^b \operatorname{Im} f(t)\, dt = r \int_a^b |f(t)|\, dt$$

and

$$\varphi \int_a^b \operatorname{Re} f(t)\,dt - \psi \int_a^b \operatorname{Im} f(t)\,dt = \left| \int_a^b f(t)\,dt \right|.$$

References

[1] E. BERKSON, Some types of Banach spaces, Hermitian systems and Bade functionals, *Trans. Amer. Math. Soc.,* **106**(1965), 376-385.

[2] S.S. DRAGOMIR, *Semi-Inner Products and Applications*, Nova Science Publishers Inc., New York, 2004, pp. 222.

[3] S.S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Preprint. [Online http://front.math.ucdavis.edu/math.FA/0309354].

[4] S.S. DRAGOMIR, Reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in Hilbert spaces, *RGMIA Res. Rep. Coll.,* **7**(2004), Supplement, Article 11 [Online http://rgmia.vu.edu.au/v7(E).html].

[5] S. GUDDER and D. STRAWTHER, Strictly convex normed linear spaces, *Proc. Amer. Math. Soc.,* **59**(2)(1976), 263-267.

[6] J. KARAMATA, *Stieltjes Integral, Theory and Practice* (Serbo-Croatian) SANU, Posebna Izdonija, 154, Beograd, 1949.

[7] G. LUMER, Semi-inner product spaces, *Trans. Amer. Math. Soc.,* **100**(1961), 29-43.

[8] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[9] B. NATH, On a generalisation of semi-inner product spaces, *Math. J. Okoyama Univ.,* **15**(10)(1971), 1-6.

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