A REVERSE OF THE GENERALISED TRIANGLE INEQUALITY IN NORMED SPACES AND APPLICATIONS

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Abstract. A new reverse of the generalised triangle inequality that complements the classical results of Diaz and Metcalf is obtained. Applications for inner product spaces and for complex numbers are provided.

1. Introduction

In [1], Diaz and Metcalf established the following reverse of the generalised triangle inequality in real or complex normed linear spaces.

If $F : X \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a linear functional of a unit norm defined on the normed linear space X endowed with the norm $\|\cdot\|$ and the vectors $x_1, \dots, x_n$ satisfy the condition

$$0 \leq r \leq \operatorname{Re} F(x_i), \qquad i \in \{1, \dots, n\}; \tag{1.1}$$

then

$$r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|, \tag{1.2}$$

where equality holds if and only if both

$$F\left(\sum_{i=1}^{n} x_i\right) = r \sum_{i=1}^{n} \|x_i\| \tag{1.3}$$

and

$$F\left(\sum_{i=1}^{n} x_i\right) = \left\| \sum_{i=1}^{n} x_i \right\|. \tag{1.4}$$

If $X = H$, $(H; \langle \cdot, \cdot \rangle)$ is an inner product space and $F(x) = \langle x, e \rangle$, $\|e\| = 1$, then the condition (1.1) may be replaced with the simpler assumption

$$0 \leq r \|x_i\| \leq \operatorname{Re} \langle x_i, e \rangle, \qquad i = 1, \dots, n, \tag{1.5}$$

which implies the reverse of the generalised triangle inequality (1.2). In this case the equality holds in (1.2) if and only if [1]

$$\sum_{i=1}^{n} x_i = r \left(\sum_{i=1}^{n} \|x_i\| \right) e. \tag{1.6}$$

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Let $F_1, \dots, F_m$ be linear functionals on X, each of unit norm. Let [1]

$$c = \sup_{x \neq 0} \left[\frac{\sum_{k=1}^{m} |F_k(x)|^2}{\|x\|^2} \right];$$

it then follows that $1 \leq c \leq m$. Suppose the vectors $x_1, \dots, x_k$ whenever $x_i \neq 0$, satisfy

$$0 \leq r_k \|x_i\| \leq \operatorname{Re} F_k(x_i), \qquad i = 1, \dots, n, \ k = 1, \dots, m. \tag{1.7}$$

Then [1]

$$\left(\frac{\sum_{k=1}^{m} r_k^2}{c} \right) \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|, \tag{1.8}$$

where equality holds if and only if both

$$F_k \left(\sum_{i=1}^{n} x_i \right) = r_k \sum_{i=1}^{n} \|x_i\|, \qquad k = 1, \dots, m \tag{1.9}$$

and

$$\sum_{k=1}^{m} \left[F_k \left(\sum_{i=1}^{n} x_i \right) \right]^2 = c \left\| \sum_{i=1}^{n} x_i \right\|^2. \tag{1.10}$$

If $X = H$, an inner product space, then, for $F_k(x) = \langle x, e_k \rangle$, where $\{e_k\}_{k=\overline{1,n}}$ is an orthonormal family in H, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$, $i, j \in \{1, \dots, k\}$, δ_{ij} is Kronecker delta, the condition (1.7) may be replaced by

$$0 \leq r_k \|x_i\| \leq \operatorname{Re} \langle x_i, e \rangle, \qquad i = 1, \dots, n, \ k = 1, \dots, m; \tag{1.11}$$

implying the following reverse of the generalised triangle inequality

$$\left(\sum_{k=1}^{m} r_k^2 \right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|, \tag{1.12}$$

where the equality holds if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| \right) \sum_{k=1}^{m} r_k e_k. \tag{1.13}$$

The main aim of this paper is to point out a different reverse of the triangle inequality than the one obtained by Diaz-Metcalf in (1.8). Its version in inner product spaces is analysed and applications for complex numbers are given as well.

For various inequalities related to the triangle inequality, see Chapter XVII of the book [3] and the references therein.

2. A New Reverse for m Functionals

The following result may be stated.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. If $x_i \in X \setminus \{0\}$, $i \in \{1, \dots, n\}$ are such that there exists the constant $r_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ and*

$$\operatorname{Re} F_k(x_i) \geq r_k \|x_i\| \quad \textit{for each} \ \ i \in \{1, \dots, n\} \ \textit{and} \ k \in \{1, \dots, m\}, \tag{2.1}$$

then

$$\sum_{i=1}^{n}\|x_i\| \le \frac{\left\|\sum_{k=1}^{m} F_k\right\|}{\sum_{k=1}^{m} r_k}\left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.2}$$

The case of equality holds in (2.2) if both

$$\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right) = \left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n}\|x_i\| \tag{2.3}$$

and

$$\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.4}$$

Proof. Utilising the hypothesis (2.1) and the properties of the modulus, we have

$$\begin{aligned} I &:= \left|\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right| \ge \left|\operatorname{Re}\left[\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right]\right| \\ &\ge \sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) = \sum_{k=1}^{m}\sum_{i=1}^{n}\operatorname{Re} F_k(x_i) \\ &\ge \left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n}\|x_i\|. \end{aligned} \tag{2.5}$$

On the other hand, by the continuity property of F_k, $k \in \{1, \dots, m\}$ we obviously have

$$I = \left|\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right| \le \left\|\sum_{k=1}^{m} F_k\right\|\left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.6}$$

Making use of (2.5) and (2.6), we deduce the desired inequality (2.2).

Now, if (2.3) and (2.4) are valid, then, obviously, the case of equality holds true in the inequality (2.2).

Conversely, if the case of equality holds in (2.2), then it must hold in all the inequalities used to prove (2.2). Therefore we have

$$\operatorname{Re} F_k(x_i) = r_k\|x_i\| \quad \text{for each} \quad i \in \{1, \dots, n\},\ k \in \{1, \dots, m\}; \tag{2.7}$$

$$\sum_{k=1}^{m}\operatorname{Im} F_k\left(\sum_{i=1}^{n} x_i\right) = 0 \tag{2.8}$$

and

$$\sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.9}$$

Note that, from (2.7), by summation over i and k, we get

$$\operatorname{Re}\left[\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right] = \left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n}\|x_i\|. \tag{2.10}$$

Since (2.8) and (2.10) imply (2.3), while (2.9) and (2.10) imply (2.4) hence the theorem is proved. ∎

Remark 1. *If the norms $\|F_k\|$, $k \in \{1, \ldots, m\}$ are easier to find, then, from (2.2), one may get the (coarser) inequality that might be more useful in practice:*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\sum_{k=1}^{m} \|F_k\|}{\sum_{k=1}^{m} r_k} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.11}$$

The case of inner product spaces, in which we may provide a simpler condition for equality, is of interest in applications.

Theorem 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i \in H \setminus \{0\}$, $k \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$. If $r_k \geq 0$, $k \in \{1, \ldots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ satisfy*

$$\operatorname{Re} \langle x_i, e_k \rangle \geq r_k \|x_i\| \quad \text{for each} \quad i \in \{1, \ldots, n\} \quad \text{and } k \in \{1, \ldots, m\}, \tag{2.12}$$

then

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\left\| \sum_{k=1}^{m} e_k \right\|}{\sum_{k=1}^{m} r_k} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.13}$$

The case of equality holds in (2.13) if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} r_k}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \left(\sum_{i=1}^{n} \|x_i\| \right) \sum_{k=1}^{m} e_k. \tag{2.14}$$

Proof. By the properties of inner product and by (2.12), we have

$$\begin{aligned} \left| \left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right| &\geq \left| \sum_{k=1}^{m} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e_k \right\rangle \right| \geq \sum_{k=1}^{m} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e_k \right\rangle \\ &= \sum_{k=1}^{m} \sum_{i=1}^{n} \operatorname{Re} \langle x_i, e_k \rangle \geq \left(\sum_{k=1}^{m} r_k \right) \sum_{i=1}^{n} \|x_i\| > 0. \end{aligned} \tag{2.15}$$

Observe also that, by (2.15), $\sum_{k=1}^{m} e_k \neq 0$.

On utlising Schwarz's inequality in the inner product space $(H; \langle \cdot, \cdot \rangle)$ for $\sum_{i=1}^{n} x_i$, $\sum_{k=1}^{m} e_k$, we have

$$\left\| \sum_{i=1}^{n} x_i \right\| \left\| \sum_{k=1}^{m} e_k \right\| \geq \left| \left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right|. \tag{2.16}$$

Making use of (2.15) and (2.16), we can conclude that (2.13) holds.

Now, if (2.14) holds true, then, by taking the norm, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} x_i \right\| &= \frac{\left(\sum_{k=1}^{m} r_k \right) \sum_{i=1}^{n} \|x_i\|}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \left\| \sum_{k=1}^{m} e_k \right\| \\ &= \frac{\left(\sum_{k=1}^{m} r_k \right)}{\left\| \sum_{k=1}^{m} e_k \right\|} \left\| \sum_{i=1}^{n} x_i \right\|, \end{aligned}$$

i.e., the case of equality holds in (2.13).

Conversely, if the case of equality holds in (2.13), then it must hold in all the inequalities used to prove (2.13). Therefore, we have

$$\operatorname{Re} \langle x_i, e_k \rangle = r_k \|x_i\| \quad \text{for each} \quad i \in \{1, \ldots, n\} \quad \text{and } k \in \{1, \ldots, m\}, \tag{2.17}$$

$$\left\|\sum_{i=1}^{n} x_i\right\| \left\|\sum_{k=1}^{m} e_k\right\| = \left|\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle\right| \tag{2.18}$$

and

$$\operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = 0. \tag{2.19}$$

From (2.17), on summing over i and k, we get

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\| . \tag{2.20}$$

By (2.19) and (2.20), we have

$$\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\| . \tag{2.21}$$

On the other hand, by the use of the following identity in inner product spaces

$$\left\| u - \frac{\langle u, v\rangle v}{\|v\|^2} \right\|^2 = \frac{\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2}{\|v\|^2}, \quad v \neq 0,$$

the relation (2.18) holds if and only if

$$\sum_{i=1}^{n} x_i = \frac{\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \sum_{k=1}^{m} e_k. \tag{2.22}$$

Finally, on utilising (2.21) and (2.22), we deduce that the condition (2.14) is necessary for the equality case in (2.13). ∎

Before we give a corollary of the above theorem, we need to state the following lemma that has been basically obtained in [2]. For the sake of completeness, we provide a short proof here as well.

Lemma 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$ such that:*

$$\|x - a\| \leq r < \|a\| . \tag{2.23}$$

Then we have the inequality

$$\|x\| \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\langle x, a\rangle \tag{2.24}$$

or, equivalently

$$\|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 \leq r^2 \|x\|^2 . \tag{2.25}$$

The case of equality holds in (2.24) (or in (2.25)) if and only if

$$\|x - a\| = r \quad \text{and} \quad \|x\|^2 + r^2 = \|a\|^2 . \tag{2.26}$$

Proof. From the first part of (2.23), we have

$$\|x\|^2 + \|a\|^2 - r^2 \leq 2\operatorname{Re}\langle x, a\rangle . \tag{2.27}$$

By the second part of (2.23) we have $\left(\|a\|^2 - r^2\right)^{\frac{1}{2}} > 0$, therefore, by (2.27), we may state that

$$0 < \frac{\|x\|^2}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \frac{2\operatorname{Re}\langle x, a\rangle}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}}. \tag{2.28}$$

Utilising the elementary inequality

$$\frac{1}{\alpha}q + \alpha p \geq 2\sqrt{pq}, \quad \alpha > 0,\ p > 0,\ q \geq 0;$$

with equality if and only if $\alpha = \sqrt{\frac{q}{p}}$, we may state (for $\alpha = \left(\|a\|^2 - r^2\right)^{\frac{1}{2}}$, $p = 1$, $q = \|x\|^2$) that

$$2\|x\| \leq \frac{\|x\|^2}{\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2\right)^{\frac{1}{2}}. \tag{2.29}$$

The inequality (2.24) follows now by (2.28) and (2.29).

From the above argument, it is clear that the equality holds in (2.24) if and only if it holds in (2.28) and (2.29). However, the equality holds in (2.28) if and only if $\|x - a\| = r$ and in (2.29) if and only if $\left(\|a\|^2 - r^2\right)^{\frac{1}{2}} = \|x\|$.

The proof is thus completed. ∎

We may now state the following corollary.

Corollary 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $e_k, x_i \in H\backslash\{0\}$, $k \in \{1,\ldots,m\}$, $i \in \{1,\ldots,n\}$. If $\rho_k \geq 0$, $k \in \{1,\ldots,m\}$ with*

$$\|x_i - e_k\| \leq \rho_k < \|e_k\| \text{ for each } i \in \{1,\ldots,n\} \text{ and } k \in \{1,\ldots,m\}, \tag{2.30}$$

then

$$\sum_{i=1}^{n}\|x_i\| \leq \frac{\left\|\sum_{k=1}^{m} e_k\right\|}{\sum_{k=1}^{m}\left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.31}$$

The case of equality holds in (2.31) if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m}\left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}}{\left\|\sum_{k=1}^{m} e_k\right\|^2}\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m} e_k.$$

Proof. Utilising Lemma 1, we have from (2.30) that

$$\|x_i\|\left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\langle x_i, e_k\rangle$$

for each $k \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,n\}$.

Applying Theorem 2 for

$$r_k := \left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}, \quad k \in \{1, \dots, m\},$$

we deduce the desired result. ∎

Remark 2. *If* $\{e_k\}_{k\in\{1,\dots,m\}}$ *are orthogonal, then (2.31) becomes*

$$\sum_{i=1}^{n} \|x_i\| \le \frac{\left(\sum_{k=1}^{m} \|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^{m} \left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}} \left\|\sum_{i=1}^{n} x_i\right\| \tag{2.32}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} \left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}}{\sum_{k=1}^{m} \|e_k\|^2} \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} e_k.$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ *is assumed to be orthonormal and*

$$\|x_i - e_k\| \le \rho_k \quad \text{for } k \in \{1, \dots, m\}, \ i \in \{1, \dots, n\}$$

where $\rho_k \in [0, 1)$ *for* $k \in \{1, \dots, m\}$, *then*

$$\sum_{i=1}^{n} \|x_i\| \le \frac{\sqrt{m}}{\sum_{k=1}^{m} (1 - \rho_k^2)^{\frac{1}{2}}} \left\|\sum_{i=1}^{n} x_i\right\| \tag{2.33}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} (1 - \rho_k^2)^{\frac{1}{2}}}{m} \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} e_k.$$

The following lemma may be stated as well [2].

Lemma 2. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $x, y \in H$ *and* $M \ge m > 0$. *If*

$$\operatorname{Re}\langle My - x, x - my\rangle \ge 0 \tag{2.34}$$

or, equivalently,

$$\left\|x - \frac{m+M}{2} y\right\| \le \frac{1}{2}(M - m)\|y\|, \tag{2.35}$$

then

$$\|x\|\|y\| \le \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x, y\rangle. \tag{2.36}$$

The equality holds in (2.36) if and only if the case of equality holds in (2.34) and

$$\|x\| = \sqrt{mM}\|y\|. \tag{2.37}$$

Proof. Obviously,

$$\operatorname{Re}\langle My - x, x - my\rangle = (M+m)\operatorname{Re}\langle x, y\rangle - \|x\|^2 - mM\|y\|^2.$$

Then (2.34) is clearly equivalent to

$$\frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\|y\|^2 \le \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x, y\rangle. \tag{2.38}$$

Since, obviously,

$$2\|x\|\|y\| \leq \frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\|y\|^2, \tag{2.39}$$

with equality iff $\|x\| = \sqrt{mM}\|y\|$, hence (2.38) and (2.39) imply (2.36).

The case of equality is obvious and we omit the details. ∎

Finally, we may state the following corollary of Theorem 2.

Corollary 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i \in H\backslash\{0\}$, $k \in \{1,\dots,m\}$, $i \in \{1,\dots,n\}$. If $M_k > \mu_k > 0$, $k \in \{1,\dots,m\}$ are such that either*

$$\operatorname{Re}\langle M_k e_k - x_i, x_i - \mu_k e_k\rangle \geq 0 \tag{2.40}$$

or, equivalently,

$$\left\|x_i - \frac{M_k + \mu_k}{2} e_k\right\| \leq \frac{1}{2}(M_k - \mu_k)\|e_k\|$$

for each $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,n\}$, then

$$\sum_{i=1}^n \|x_i\| \leq \frac{\left\|\sum_{k=1}^m e_k\right\|}{\sum_{k=1}^m \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\|}\left\|\sum_{i=1}^n x_i\right\|. \tag{2.41}$$

The case of equality holds in (2.41) if and only if

$$\sum_{i=1}^n x_i = \frac{\sum_{k=1}^m \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\|}{\left\|\sum_{k=1}^m e_k\right\|^2}\sum_{i=1}^n \|x_i\|\sum_{k=1}^m e_k.$$

Proof. Utilising Lemma 2, by (2.40) we deduce

$$\|x_i\|\frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\| \leq \operatorname{Re}\langle x_i, e_k\rangle$$

for each $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,n\}$.

Applying Theorem 2 for

$$r_k := \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k}\|e_k\|, \quad k \in \{1,\dots,m\},$$

we deduce the desired result. ∎

3. Applications for Complex Numbers

Let $\mathbb{C}$ be the field of complex numbers. If $z = \operatorname{Re} z + i\operatorname{Im} z$, then by $|\cdot|_p : \mathbb{C} \to [0,\infty)$, $p \in [1,\infty]$ we define the *$p-$modulus* of z as

$$|z|_p := \begin{cases} \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} & \text{if } p = \infty, \\ \\ (|\operatorname{Re} z|^p + |\operatorname{Im} z|^p)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \end{cases}$$

where $|a|$, $a \in \mathbb{R}$ is the usual modulus of the real number a.

For $p = 2$, we recapture the usual modulus of a complex number, i.e.,

$$|z|_2 = \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2} = |z|, \quad z \in \mathbb{C}.$$

It is well known that $\left(\mathbb{C}, |\cdot|_p\right)$, $p \in [1, \infty]$ is a Banach space over the complex number field $\mathbb{C}$.

Consider the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F : \mathbb{C} \to \mathbb{C}$, $F(z) = az$ with $a \in \mathbb{C}$, $a \neq 0$. Obviously, F is linear on $\mathbb{C}$. For $z \neq 0$, we have

$$\frac{|F(z)|}{|z|_1} = \frac{|a|\,|z|}{|z|_1} = \frac{|a| \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{|\operatorname{Re} z| + |\operatorname{Im} z|} \leq |a|.$$

Since, for $z_0 = 1$, we have $|F(z_0)| = |a|$ and $|z_0|_1 = 1$, hence

$$\|F\|_1 := \sup_{z \neq 0} \frac{|F(z)|}{|z|_1} = |a|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_1)$ and $\|F\|_1 = |a|$.

We can apply Theorem 1 to state the following reverse of the generalised triangle inequality for complex numbers.

Proposition 1. *Let a_k, $x_j \in \mathbb{C}$, $k \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. If there exist the constants $r_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k [|\operatorname{Re} x_j| + |\operatorname{Im} x_j|] \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j \tag{3.1}$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then

$$\sum_{j=1}^n [|\operatorname{Re} x_j| + |\operatorname{Im} x_j|] \leq \frac{\left|\sum_{k=1}^m a_k\right|}{\sum_{k=1}^m r_k} \left[\left|\sum_{j=1}^n \operatorname{Re} x_j\right| + \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right]. \tag{3.2}$$

The case of equality holds in (3.2) if both

$$\begin{aligned} &\operatorname{Re}\left(\sum_{k=1}^m a_k\right) \operatorname{Re}\left(\sum_{j=1}^n x_j\right) - \operatorname{Im}\left(\sum_{k=1}^m a_k\right) \operatorname{Im}\left(\sum_{j=1}^n x_j\right) \\ &= \left(\sum_{k=1}^m r_k\right) \sum_{j=1}^n [|\operatorname{Re} x_j| + |\operatorname{Im} x_j|] \\ &= \left|\sum_{k=1}^m a_k\right| \left[\left|\sum_{j=1}^n \operatorname{Re} x_j\right| + \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right]. \end{aligned}$$

The proof follows by Theorem 1 applied for the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F_k(z) = a_k z$, $k \in \{1, \dots, m\}$ on taking into account that:

$$\left\|\sum_{k=1}^m F_k\right\|_1 = \left|\sum_{k=1}^m a_k\right|.$$

Now, consider the Banach space $(\mathbb{C}, |\cdot|_\infty)$. If $F(z) = dz$, then for $z \neq 0$ we have

$$\frac{|F(z)|}{|z|_\infty} = \frac{|d|\,|z|}{|z|_\infty} = \frac{|d| \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}} \leq \sqrt{2}\,|d|.$$

Since, for $z_0 = 1 + i$, we have $|F(z_0)| = \sqrt{2}\,|d|$, $|z_0|_\infty = 1$, hence

$$\|F\|_\infty := \sup_{z \neq 0} \frac{|F(z)|}{|z|_\infty} = \sqrt{2}\,|d|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_\infty)$ and $\|F\|_\infty = \sqrt{2}\,|d|$.

If we apply Theorem 1, then we can state the following reverse of the generalised triangle inequality for complex numbers.

Proposition 2. *Let $a_k, x_j \in \mathbb{C}$, $k \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. If there exist the constants $r_k \geq 0$, $k \in \{1,\dots,m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then

$$\sum_{j=1}^n \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \sqrt{2} \cdot \frac{|\sum_{k=1}^m a_k|}{\sum_{k=1}^m r_k} \max\left\{\left|\sum_{j=1}^n \operatorname{Re} x_j\right|, \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right\}. \tag{3.3}$$

The case of equality holds in (3.3) if both

$$\begin{aligned}
&\operatorname{Re}\left(\sum_{k=1}^m a_k\right)\operatorname{Re}\left(\sum_{j=1}^n x_j\right) - \operatorname{Im}\left(\sum_{k=1}^m a_k\right)\operatorname{Im}\left(\sum_{j=1}^n x_j\right) \\
&= \left(\sum_{k=1}^m r_k\right)\sum_{j=1}^n \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \\
&= \sqrt{2}\left|\sum_{k=1}^m a_k\right| \max\left\{\left|\sum_{j=1}^n \operatorname{Re} x_j\right|, \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right\}.
\end{aligned}$$

Finally, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$.

Let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = cz$. By Hölder's inequality, we have

$$\frac{|F(z)|}{|z|_{2p}} = \frac{|c|\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\left(|\operatorname{Re} z|^{2p} + |\operatorname{Im} z|^{2p}\right)^{\frac{1}{2p}}} \leq 2^{\frac{1}{2} - \frac{1}{2p}}\,|c|.$$

Since, for $z_0 = 1 + i$ we have $|F(z_0)| = 2^{\frac{1}{2}}\,|c|$, $|z_0| = 2^{\frac{1}{2p}}$ $(p \geq 1)$, hence

$$\|F\|_{2p} := \sup_{z \neq 0} \frac{|F(z)|}{|z|_{2p}} = 2^{\frac{1}{2} - \frac{1}{2p}}\,|c|,$$

showing that F is a bounded linear functional on $\left(\mathbb{C}, |\cdot|_{2p}\right)$, $p \geq 1$ and $\|F\|_{2p} = 2^{\frac{1}{2} - \frac{1}{2p}}\,|c|$.

If we apply Theorem 1, then we can state the following proposition.

Proposition 3. *Let $a_k, x_j \in \mathbb{C}$, $k \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. If there exist the constants $r_k \geq 0$, $k \in \{1,\dots,m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p}\right]^{\frac{1}{2p}} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$\sum_{j=1}^{n} \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \leq 2^{\frac{1}{2} - \frac{1}{2p}} \frac{\left|\sum_{k=1}^{m} a_k\right|}{\sum_{k=1}^{m} r_k} \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|^{2p} + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}} . \tag{3.4}$$

The case of equality holds in (3.4) if both:

$$\begin{aligned} &\operatorname{Re}\left(\sum_{k=1}^{m} a_k\right) \operatorname{Re}\left(\sum_{j=1}^{n} x_j\right) - \operatorname{Im}\left(\sum_{k=1}^{m} a_k\right) \operatorname{Im}\left(\sum_{j=1}^{n} x_j\right) \\ &= \left(\sum_{k=1}^{m} r_k\right) \sum_{j=1}^{n} \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \\ &= 2^{\frac{1}{2} - \frac{1}{2p}} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|^{2p} + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}} . \end{aligned}$$

Remark 3. *If in the above proposition we choose* $p = 1$, *then we have the following reverse of the generalised triangle inequality for complex numbers*

$$\sum_{j=1}^{n} |x_j| \leq \frac{\left|\sum_{k=1}^{m} a_k\right|}{\sum_{k=1}^{m} r_k} \left| \sum_{j=1}^{n} x_j \right|$$

provided $x_j, a_k,\ j \in \{1, \dots, n\},\ k \in \{1, \dots, m\}$ *satisfy the assumption*

$$r_k |x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1, \dots, n\},\ k \in \{1, \dots, m\}$. *Here* $|\cdot|$ *is the usual modulus of a complex number and* $r_k > 0,\ k \in \{1, \dots, m\}$ *are given.*

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