A Short Note on the Erdös – Debrunner Inequality

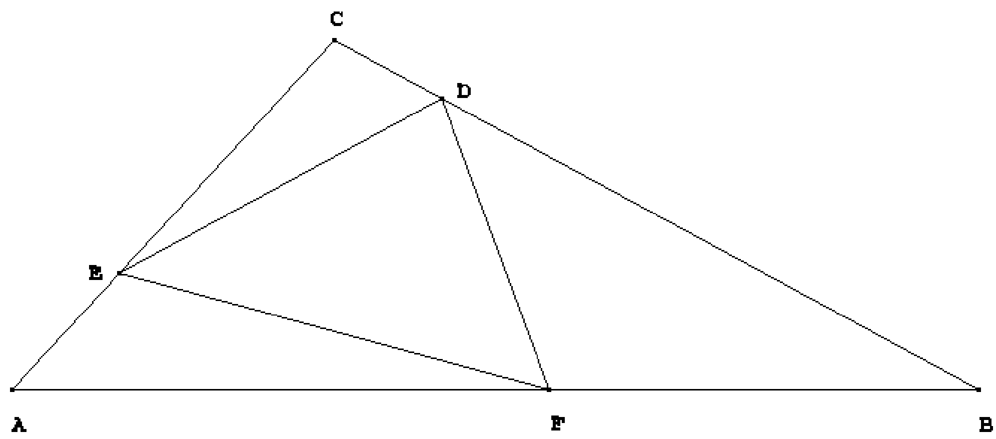
Walther JANOUS (Innsbruck, Austria)

Abstract. In this note the power-mean generalization of the Erdös – Debrunner inequality is reconsidered and bounds for the maximum order p_{max} of such an inequality are obtained.

Introduction.

Let ABC be an arbitrary triangle and D, E and F be arbitrary points on sides BC, CA and AB resp., all three being different from the vertices of ABC.

Then triangle ABC is divided into four smaller triangles, a central one DEF, and three corner ones AEF, BDF and CED.



Figure

Let F_1, F_2 and F_3 be the areas of the three corner triangles and F_0 be the one of the central triangle.

Then the *Erdös-Debrunner inequality* says

(1) $F_0 \geq \min(F_1, F_2, F_3)$

where equality occurs if and only if D, E and F are the midpoints of the respective sides. (Cf. [1], p. 81, for an extensive list of references concerning this inequality.)

Speaking in the language of power-means inequality (1) reads

$$F_0 \geq M_{-\infty}(F_1, F_2, F_3)$$

where the p-th power-mean of three positive real numbers x, y and z is defined by

$$M_p(x, y, z) = \begin{cases} \left(\dfrac{x^p + y^p + z^p}{3}\right)^{1/p} & , p \neq 0 \\ \sqrt[3]{xyz} & , p = 0 \end{cases}$$

Then $M_p(x, y, z)$ is (weakly) increasing as p increases and $M_{-\infty}(x, y, z) = \lim\limits_{p \to -\infty} M_p(x, y, z) = \min(x, y, z)$.

Therefore it is natural to ask whether or not there do exist inequalities of type

(2) $F_0 \geq M_p(F_1, F_2, F_3)$

where $p > -\infty$.

Subsequently we will show that this is indeed so and we will give a bound for the maximum value p_{max} of p. Thereby we also will falsify a result stated and "proved" in [2].

At the end of this note we shall state two conjectures for further research.

Bounds for p_{max} in inequality (2).

Before stating the announced result we introduce the method of proof frequently applied in situations as the present one.

Let BC, CA and AB be divided by D, E and F in ratios $t : (1 - t)$, $u : (1 - u)$ and $v : (1 - v)$, resp., where $0 < t, u, v < 1$.

Then $F_1 = (1 - u)\cdot v\cdot F_\Delta$, $F_2 = (1 - v)\cdot t\cdot F_\Delta$ and $F_3 = (1 - t)\cdot u\cdot F_\Delta$, where F_Δ denotes the area of triangle ABC. (Note for instance for F_1: $AF = v\cdot AB$ and $AE = (1 - u)\cdot AC$.)

Therefore $F_0 = F_\Delta - F_1 - F_1 - F_3$ becomes $F_0 = (t\cdot u\cdot v + (1 - t)\cdot(1 - u)\cdot(1 - v))\cdot F_\Delta$.

Furthermore, $\frac{F_0}{F_1} = \frac{1 - t - u - v + tu + tv + uv}{(1 - u)v}$, that is $\frac{F_0}{F_1} = \frac{1-t}{v} + \frac{t}{1-u} - 1$.

As we get similar expressions for $\frac{F_0}{F_2}$ and $\frac{F_0}{F_3}$ we introduce the notation $x = \frac{t}{1-u}$, $y = \frac{u}{1-v}$ and $z = \frac{v}{1-t}$ yielding $\frac{F_0}{F_1} = \frac{1}{z} + x - 1$, $\frac{F_0}{F_2} = \frac{1}{x} + y - 1$ and $\frac{F_0}{F_3} = \frac{1}{y} + z - 1$.

We now show that p has to be negative in order that inequality (2) holds in general.

Indeed, let $p = 0$. Then for (2) there had to hold $\frac{F_0}{F_1} \cdot \frac{F_0}{F_2} \cdot \frac{F_0}{F_3} \geq 1$.

But $t = \frac{1}{2}$, $u = \frac{1}{3}$ and $v = \frac{2}{3}$ lead to the contradiction $\frac{8}{9} \geq 1$.

Therefore we let $p = -q$, where $q > 0$, and thus obtain for (2) the equivalent inequality

$$F_1^{-q} + F_2^{-q} + F_3^{-q} \geq 3\cdot F_0^{-q}\text{, that is}$$

$$\left(\frac{F_0}{F_1}\right)^q + \left(\frac{F_0}{F_2}\right)^q + \left(\frac{F_0}{F_3}\right)^q \geq 3\text{, i.e.}$$

(3) $$\left(\frac{1}{z} + x - 1\right)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + z - 1\right)^q \geq 3$$

where of course x, y, $z > 0$ have to satisfy $\frac{1}{z} + x - 1 \geq 0$, $\frac{1}{x} + y - 1 \geq 0$ and $\frac{1}{y} + z - 1 \geq 0$.

We are now in the position to state and show the following

THEOREM. p_{max} for inequality (2) satisfies $-1 \le p_{max} \le -\frac{ln(3/2)}{ln 2}$.

PROOF. In order to prove this assertion we have to show that the minimum-value q_{min} such that inequality (3) is true in general fulfills $\frac{ln(3/2)}{ln 2} \le q_{min} \le 1$.

i) $q_{min} \le 1$. Indeed, (3) becomes for q = 1: $x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z} \ge 6$.

But this inequality follows from $t+\frac{1}{t} \ge 2$ whenever t > 0.

ii) $q_{min} \ge \frac{ln(3/2)}{ln 2}$. We let t = $\frac{1}{2}$, and v = 1 – u (0 < u < 1).

Then $\frac{F_0}{F_1} = \frac{u}{1-u}$ and $\frac{F_0}{F_2} = \frac{F_0}{F_3} = 2(1-u)$, whence inequality (3) reads

$$\left(\frac{u}{1-u}\right)^q + 2\cdot(2(1-u))^q \ge 3 \text{ where } 0 < u < 1.$$

Since the expression on the left-hand side of this inequality is continuous as u → 0 we arrive at $2 \cdot 2^q \ge 3$ and the proof is complete.

Remark. In [2] it is "shown" by an erroneous argument that p_{max} equals $-\frac{1}{3}$ contradicting the inequality $p_{max} \le -\frac{ln(3/2)}{ln 2} = -0.58...$.

Two conjectures.

At the end of this note we state two conjectures. (The second one of them is very likely to be settled by nonelementary means only.)

CONJECTURE 1. Let x, y and z be positive real numbers such that $\frac{1}{z}+x-1 \ge 0$, $\frac{1}{x}+y-1 \ge 0$ and $\frac{1}{y}+z-1 \ge 0$.

Then for any q > 0 the minimum of the left-hand expression in (3) is attained at x, y and z satisfying x·y·z = 1.

CONJECTURE 2. In the theorem there holds $p_{max} = -\frac{ln(3/2)}{ln 2}$.

References.

[1] O. Bottema et al., *Geometric Inequalities.* Wolters and Noordhoff Groningen 1969.
[2] D. Mavlo, Solution of Problem 4 (posed by himself) [in Bulgarian]. Obuch. po matem. i inform. **XIV** (1989), nr. 3, 48-51.

Ursulinengymnasium
Fürstenweg 86
A 6020 Innsbruck
Austria
e-mail: walther.janous@tirol.com