AN APPROXIMATION OF HANKEL TRANSFORM FOR THE FUNCTIONS OF BOUNDED VARIATION

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Abstract. Using some techniques developed by S.S. Dragomir in connection to Ostrowski's integral inequality, we point out some approximation results for the Hankel Transform of functions of bounded variation.

1. Introduction

Two-dimensional systems may often show circular symmetry, for example optical systems are often constructed from components that, in themselves, are circulary symmetrical.

When circular symmetry exists, that is, when $f(x,y) = f(r)$, $r^2 = x^2 + y^2$, then the bidimensional Fourier transform can be represented in the following way [2, p. 244 - p. 250]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(xu+yv)}dxdy \tag{1.1}$$

$$= \int_0^{\infty}\int_0^{2\pi} f(r)e^{-i2\pi qr\cos(\theta-\varphi)}rdrd\theta = \int_0^{\infty} f(r)\left[\int_0^{2\pi} e^{-i2\pi qr\cos(\theta-\varphi)}d\theta\right]rdr$$

$$= 2\pi\int_0^{\infty} f(r)J_0(2\pi qr)rdr$$

where $x + iy = re^{i\theta}$, $u + iv = qe^{i\varphi}$, $q^2 = u^2 + v^2$ and we have used the relation

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos\beta}d\beta. \tag{1.2}$$

We refer to $G(f)(q)$ given by

$$G(f)(q) = 2\pi\int_0^{\infty} f(r)J_0(2\pi qr)rdr \tag{1.3}$$

as the Hankel transform (of zero order) of $f(r)$.

The main aim of the present article is to point out some estimates of the Hankel transform for functions of bounded variation on an finite interval $[a,b]$ by the use of some techniques developed by S.S. Dragomir [6] in connection to Ostrowski's integral inequality. Some adaptive quadrature formulae will be also derived.

For recent results in approximating the Hankel transform of absolutely continuous functions on using Ostrowsky type inequalities, see [3].

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2. An Integral Representation

Let $J_1(\cdot)$ denote the first-order Bessel function of the first kind. That is,

$$J_1(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\varphi - i\varphi},\ x\in\mathbb{R}. \tag{2.1}$$

We define the corresponding *Bessel's mean* as:

$$B_1(z,w) := \frac{zJ_1(z) - wJ_1(w)}{z-w} \text{ if } w\neq z,\ w,z\in\mathbb{C}. \tag{2.2}$$

The following representation of the Hankel transform for mappings of bounded variation holds.

Theorem 1. *Let $g:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ be of bounded variation on $[a,b]$. Then we have the representation*

$$\begin{aligned} G(g)(\rho) &= \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho)\times\int_a^b g(s)\,ds \\ &\quad+\frac{2\pi}{b-a}\int_a^b A(g)(r)\,rJ_0(2\pi r\rho)\,dr, \end{aligned} \tag{2.3}$$

where the kernel $A(g):[a,b]\to\mathbb{K}$ is given by the Riemann Stieltjes integral

$$A(g)(r) := \int_a^r (s-a)\,dg(s) + \int_r^b (s-b)\,dg(s), \tag{2.4}$$

for all $\rho\in[a,b]$, $\rho\neq 0$.

Proof. Using the integration by parts formula for the Riemann-Stieltjes integral (see also [6]), we have that

$$\int_a^x (s-a)\,dg(s) = (x-a)\,g(x) - \int_a^x g(s)\,ds \tag{2.5}$$

and

$$\int_x^b (s-b)\,dg(s) = (b-x)\,g(x) - \int_x^b g(s)\,ds, \tag{2.6}$$

for all $x\in[a,b]$.
Adding (2.5) and (2.6), and diving by $(b-a)$, we get the representation [6]

$$g(x) = \frac{1}{b-a}\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^x (s-a)\,dg(s) + \frac{1}{b-a}\int_x^b (s-b)\,dg(s), \tag{2.7}$$

for all $x\in[a,b]$.
Now, using the definition of Hankel transform, we have:

$$G(g)(\rho) = 2\pi\int_a^b g(r)\,rJ_0(2\pi\rho r)\,dr \tag{2.8}$$

$$\begin{aligned} &= 2\pi \int_a^b \left[\frac{1}{b-a} \int_a^b g(s)\, ds + \frac{1}{b-a} \int_a^r (s-a)\, dg(s) \right. \\ &\quad \left. + \frac{1}{b-a} \int_r^b (s-b)\, dg(s) \right] r J_0(2\pi\rho r)\, dr \\ &= 2\pi \frac{1}{b-a} \int_a^b g(s)\, ds \times \int_a^b r J_0(2\pi\rho r)\, dr + \frac{2\pi}{b-a} \int_a^b A(g)(r)\, r J_0(2\pi\rho r)\, dr. \end{aligned}$$

Let us compute $\int_a^b r J_0(2\pi\rho r)\, dr$.
Consider the change of variable $r' = 2\pi r\rho$. Then $r = \frac{r'}{2\pi\rho}$, $dr = \frac{dr'}{2\pi\rho}$ and

$$\begin{aligned} \int_a^b r J_0(2\pi\rho r)\, dr &= \int_{2\pi a\rho}^{2\pi b\rho} \frac{r'}{(2\pi\rho)^2} J_0(r')\, dr' \\ &= \frac{1}{(2\pi\rho)^2} \int_{2\pi a\rho}^{2\pi b\rho} r' J_0(r')\, dr'. \end{aligned}$$

It is a well known property of Bessel functions that

$$\int_0^x \xi J_0(\xi)\, d\xi = x J_1(x),\ x \in \mathbb{R}, \tag{2.9}$$

where J_1 is the first-order Bessel function defined above.
Hence we obtain

$$\begin{aligned} \int_{2\pi a\rho}^{2\pi b\rho} r' J_0(r')\, dr' &= \int_0^{2\pi b\rho} r' J_0(r')\, dr' - \int_0^{2\pi a\rho} r' J_0(r')\, dr' \\ &= 2\pi b\rho J_1(2\pi b\rho) - 2\pi a\rho J_1(2\pi a\rho). \end{aligned}$$

Consequently, we can state that

$$\int_a^b r J_0(2\pi\rho r)\, dr = \frac{1}{(2\pi\rho)^2} \left[2\pi b\rho J_1(2\pi b\rho) - 2\pi a\rho J_1(2\pi a\rho)\right]$$

and

$$2\pi \frac{1}{b-a} \int_a^b g(s)\, ds \times \int_a^b r J_0(2\pi\rho r)\, dr = \frac{1}{\rho} B_1(2\pi b\rho, 2\pi a\rho).$$

Using (2.7), we deduce the desired representation (2.3). ∎

In practical applications, we have $a = 0$ and $b = 1$. Consequently, we can state the following corollary.

Corollary 1. *Let $g : [0,1] \to \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ be a mapping of bounded variation on $[0,1]$. Then we have the representation*

$$G(g)(\rho) = \frac{J_1(2\pi\rho)}{\rho} \times \int_0^1 g(s)\, ds + 2\pi \int_0^1 \tilde{A}(g)(r)\, r J_0(2\pi r\rho)\, dr, \tag{2.10}$$

where $\tilde{A}(g) : [0,1] \to \mathbb{K}$ is given by

$$\tilde{A}(g)(r) := \int_0^r s dg(s) + \int_r^1 (s-1)\, dg(s),$$

for all $\rho \in [0,1]$.

Now, let us define the mapping with real values $I(g):[a,b]\to[0,\infty)$ given by

$$I(g)(\rho):=|G(g)(\rho)|^2,\ \rho\in[a,b]. \tag{2.11}$$

Using the well known property of complex numbers

$$|x+y|^2=|x|^2+2\operatorname{Re}(x\bar{y})+|y|^2,\ \text{for any } x,y\in\mathbb{C}, \tag{2.12}$$

we can state the following corollary:

Corollary 2. *With the assumptions from Theorem 1, we have*

$$\begin{aligned} & I(g)(\rho) \\ = & \frac{1}{\rho^2}|B_1(2\pi b\rho,2\pi a\rho)|^2\left|\int_a^b g(s)\,ds\right|^2 \\ & +4\frac{\pi}{\rho(b-a)}\operatorname{Re}\left[B_1(2\pi b\rho,2\pi a\rho)\int_a^b g(s)\,ds\times\int_a^b\overline{A(g)\,rJ_0(2\pi r\rho)}dr\right] \\ & +4\frac{\pi^2}{(b-a)^2}\left|\int_a^b A(g)\,rJ_0(2\pi\rho r)\,dr\right|^2, \end{aligned} \tag{2.13}$$

for all $\rho\in[a,b],\ \rho\neq0.$
If $a=0,\ b=1,$ *then*

$$\begin{aligned} & I(g)(\rho) \\ = & \frac{1}{\rho^2}|J_1(2\pi\rho)|^2\left|\int_0^1 g(s)\,ds\right|^2 \\ & +\frac{4\pi}{\rho}\operatorname{Re}\left[J_1(2\pi\rho)\int_0^1 g(s)\,ds\times\int_0^1\tilde{A}(g)(r)\,rJ_0(2\pi r\rho)\,dr\right]. \end{aligned} \tag{2.14}$$

3. Integral Inequalities

The main aim of this section is to point out an estimate for the remainder

$$R(g)(\rho):=\frac{2\pi}{b-a}\int_a^b A(g)\,rJ_0(2\pi\rho r)\,dr \tag{3.1}$$

in formula (2.3).
We can state the following integral inequality.

Theorem 2. *Let* $g:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ *be of bounded variation on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned} & \left|G(g)(\rho)-\frac{1}{\rho}B_1(2\pi b\rho,2\pi a\rho)\times\int_a^b g(s)\,ds\right| \\ \leq & 2\pi\bigvee_a^b(g)\int_a^b\left[\frac{1}{2}+\frac{\left|r-\frac{a+b}{2}\right|}{b-a}\right]|r|\,dr, \end{aligned} \tag{3.2}$$

where $\bigvee_a^b(g)$ *denotes the total variation of* g *on the interval* $[a,b]$, *for all* $\rho\in[a,b]$, $\rho\neq0.$

Proof. Using the identity (2.3) , we obtain

$$\begin{aligned} &\left| G\left(g\right)\left(\rho\right) - \frac{1}{\rho} B_1\left(2\pi b\rho, 2\pi a\rho\right) \times \int_a^b g\left(s\right) ds \right| \\ \leq\ & \frac{2\pi}{b-a} \left| \int_a^b \left(\int_a^r \left(s-a\right) dg\left(s\right) \right) r J_0\left(2\pi r\rho\right) dr \right| \\ & + \frac{2\pi}{b-a} \left| \int_a^b \left(\int_r^b \left(s-b\right) dg\left(s\right) \right) r J_0\left(2\pi r\rho\right) dr \right| \\ =\ & : B\left(\rho\right), \end{aligned} \tag{3.3}$$

It is obvious that,

$$\begin{aligned} \left| J_0\left(2\pi r\rho\right) \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-i2\pi r\rho\cos\beta} d\beta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| e^{-i2\pi r\rho\cos\beta} \right| d\beta \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\beta = 1. \end{aligned}$$

On the other hand, it is well known that if $p : [c,d] \to \mathbb{K}$ is continuous and $v : [c,d] \to \mathbb{R}$ is of bounded variation on $[c,d]$, then p is Riemann-Stieltjes integrable with respect to v and

$$\left| \int_c^d p\left(x\right) dv\left(x\right) \right| \leq \sup_{x\in[c,d]} \left| p\left(x\right) \right| \bigvee_c^d \left(v\right). \tag{3.4}$$

Using this property, we have that

$$\begin{aligned} &\left| \int_a^b \left(\int_a^r \left(s-a\right) dg\left(s\right) \right) r J_0\left(2\pi r\rho\right) dr \right| \\ \leq\ & \int_a^b \left| \int_a^r \left(s-a\right) dg\left(s\right) \right| \left| r \right| \left| J_0\left(2\pi r\rho\right) \right| dr \\ \leq\ & \int_a^b \left(r-a\right) \bigvee_a^r \left(g\right) \left| r \right| dr, \end{aligned} \tag{3.5}$$

and, similarly,

$$\left| \int_a^b \left(\int_r^b \left(s-b\right) dg\left(s\right) \right) r J_0\left(2\pi r\rho\right) dr \right| \leq \int_a^b \left(b-r\right) \bigvee_b^r \left(g\right) \left| r \right| dr. \tag{3.6}$$

By the estimates (3.5) and (3.6), we can write down that

$$\begin{aligned} B(\rho) &\leq \frac{2\pi}{b-a}\int_a^b (r-a)\bigvee_a^r(g)\,|r|\,dr + \frac{2\pi}{b-a}\int_a^b (b-r)\bigvee_b^r(g)\,|r|\,dr \\ &= \frac{2\pi}{b-a}\int_a^b \left[(r-a)\bigvee_a^r(g) + (b-r)\bigvee_b^r(g)\right]|r|\,dr \\ &\leq \frac{2\pi}{b-a}\int_a^b \max(r-a,b-r)\left[\bigvee_a^r(g)+\bigvee_b^r(g)\right]|r|\,dr \\ &= \frac{2\pi}{b-a}\int_a^b \left[\frac{1}{2}(b-a)+\left|r-\frac{a+b}{2}\right|\right]\bigvee_a^b(g)\,|r|\,dr \\ &= \frac{2\pi}{b-a}\bigvee_a^b(g)\int_a^b \left[\frac{1}{2}(b-a)+\left|r-\frac{a+b}{2}\right|\right]|r|\,dr, \end{aligned}$$

and the inequality (3.2) is thus achieved. ∎

Remark 1. *In practical applications,* $a \geq 0$*, so the bound in* (3.2) *becomes*

$$2\pi\bigvee_a^b(g)\int_a^b\left[\frac{1}{2}+\frac{\left|r-\frac{a+b}{2}\right|}{b-a}\right]rdr.$$

As a simple calculation shows that

$$\int_a^b\left[\frac{1}{2}+\frac{\left|r-\frac{a+b}{2}\right|}{b-a}\right]rdr = \frac{a+b}{2}\cdot(b-a),$$

then we get the inequality

$$\left|G(g)(\rho)-\frac{1}{\rho}B_1(2\pi b\rho,2\pi a\rho)\times\int_a^b g(s)\,ds\right| \leq \pi(a+b)(b-a)\bigvee_a^b(g). \tag{3.7}$$

for all $\rho\in[a,b]$, $\rho\neq 0$.

Remark 2. *If we assume that* $a=0$ *and* $b>0$ *in* (3.7)*, then we obtain*

$$\left|G(g)(\rho)-\frac{J_1(2\pi b\rho)}{\rho}\times\int_0^b g(s)\,ds\right| \leq \pi b^2\bigvee_a^b(g). \tag{3.8}$$

The above inequality shows that

$$G(g)(\rho)\approx\frac{J_1(2\pi b\rho)}{\rho}\times\int_0^b g(s)\,ds,$$

where $b\to 0+$ *and the precision of the approximation is of order 3.*

In some practical applications the upper limit of integration is $b=1$, therefore the following corollary is required.

Corollary 3. *Let* $g:[0,1]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *be a mapping of bounded variation on* $[0,1]$*. Then we have the inequality:*

$$\left|G(g)(\rho)-\frac{J_1(2\pi\rho)}{\rho}\times\int_0^1 g(s)\,ds\right| \leq \pi\bigvee_a^b(g), \tag{3.9}$$

for all $\rho \in (0,1]$.

Remark 3. *If in Theorem 2, we assume that the mapping* $g:[0,1]\to\mathbb{R}$ *is continuously differentiable on* $[0,1]$ *and the integral* $\|g'\|_1 := \int_a^b |g'(s)|\,ds$ *is finite, then we can put in place of* $\bigvee_a^b(g)$, $\|g'\|_1$, *thus obtaining a result from the paper* [3], *i.e.,*

$$\left|G(g)(\rho) - \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho) \times \int_a^b g(s)\,ds\right| \le \pi \|g'\|_1 (a+b)(b-a),$$

for all $\rho \in [a,b]$, $\rho \neq 0$.

Using the inequality (3.2) which provides an upper bound for the remainder $R[g](\rho)$, we will point out the following inequality which approximates the mapping $I(g)(\rho)$.

Corollary 4. *Let* $g:[0,1]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *be a mapping of bounded variation on* $[0,1]$. *Then we have the inequality:*

$$\begin{aligned} &\left|I(g)(\rho) - \frac{1}{\rho^2}|B_1(2\pi b\rho, 2\pi a\rho)|^2 \left|\int_a^b g(s)\,ds\right|^2\right| \\ \le\ & \left[\frac{2}{\rho}|B_1(2\pi b\rho, 2\pi a\rho)| \left|\int_a^b g(s)\,ds\right| + E(g)(\rho)\right] E(g)(\rho) \end{aligned}$$

where

$$E(g)(\rho) := 2\pi \bigvee_a^b(g) \int_a^b \left[\frac{1}{2} + \frac{\left|r - \frac{a+b}{2}\right|}{b-a}\right]|r|\,dr,$$

for all $\rho \in [a,b]$, $\rho \neq 0$.

4. A Quadrature Formula

In this section we point out a quadrature formula for the Hankel transform

$$G(g)(\rho) = 2\pi \int_0^1 g(r)\, r J_0(2\pi\rho r)\,dr,$$

where g is assumed to be of bounded variation on $[a,b]$.

Firstly, let us assume that in formula (3.7) we have $0 \le a \le b \le 1$. Then we have the coarser upper bound

$$\left|G(g)(\rho) - \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho) \times \int_a^b g(s)\,ds\right| \le 2\pi(b-a)\bigvee_a^b(g). \tag{4.1}$$

Now, assume $I_n : 0 = x_0 < x_1 < ... < x_{n-1} < x_n$ to be a division of the interval $[a,b]$. Put $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$ the norm?? of the division. Construct the sums

$$\begin{aligned} &H(g, I_n, \rho) \\ :\ &= \frac{1}{\rho}\sum_{k=0}^{n-1} B_1(2\pi x_{i+1}\rho, 2\pi x_i\rho) \times \int_{x_i}^{x_{i+1}} g(s)\,ds \\ =\ & \frac{1}{\rho}\sum_{k=0}^{n-1}\frac{1}{h_i}\left[x_{i+1}J_1(2\pi x_{i+1}\rho) - x_i J_1(2\pi x_i\rho)\right] \times \int_{x_i}^{x_{i+1}} g(s)\,ds. \end{aligned} \tag{4.2}$$

We can state the following theorem concerning the approximation of Hankel's transform in terms of the quadrature formula (4.2).

Theorem 3. *Let $g:[0,1]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ be a mapping of bounded variation on $[0,1]$. Then we have:*

$$G(g)(\rho)=H(g,I_n,\rho)+R(g,I_n,\rho),\ \rho\in[0,1], \tag{4.3}$$

where $H(g,I_n,\rho)$ is as given by the formula (4.2) and the remainder $R(g,I_n,\rho)$ satisfies the estimate

$$|R(g,I_n,\rho)|\le 2\pi\nu(h)\bigvee_a^b(g), \tag{4.4}$$

for all $\rho\in[0,1]$.

Proof. Apply formula (4.1) on the subintervals $[x_i,x_{i+1}]$ $(i=0,...,n-1)$ to get

$$\begin{aligned}&\left|2\pi\int_{x_i}^{x_{i+1}}g(r)\,rJ_0(2\pi\rho r)\,dr-\frac{1}{\rho}B_1(2\pi x_{i+1}\rho,2\pi x_i\rho)\times\int_{x_i}^{x_{i+1}}g(s)\,ds\right|\\ \le\ & 2\pi h_i\bigvee_{x_i}^{x_{i+1}}(g).\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we obtain

$$\begin{aligned}&|R(g,I_n,\rho)|\\ \le\ &\sum_{k=0}^{n-1}\left|2\pi\int_{x_i}^{x_{i+1}}g(r)\,rJ_0(2\pi\rho r)\,dr-\frac{1}{\rho}B_1(2\pi x_{i+1}\rho,2\pi x_i\rho)\times\int_{x_i}^{x_{i+1}}g(s)\,ds\right|\\ \le\ &2\pi\sum_{k=0}^{n-1}h_i\bigvee_{x_i}^{x_{i+1}}(g)\le 2\pi\nu(h)\sum_{k=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(g)\\ =\ &2\pi\nu(h)\bigvee_a^b(g),\end{aligned}$$

and the theorem is proved. ∎

Remark 4. *Observe that if $\nu(h)\to 0$, then, by (4.4), $R(g,I_n,\rho)\to 0$ uniformly on $\rho\in(0,1]$, which proves that $H(g,I_n,\rho)$ can approximate the Hankel transform with any accuracy.*

In practical problems, the interval $[0,1]$ is divided into equidistant subintervals by the division $I_n: x_i=\frac{i}{n}$, $i=0,...,n$.
If we consider the sum

$$\begin{aligned}&H_n(g,\rho)\\ :\ =\ &\frac{1}{\rho}\sum_{k=0}^{n-1}B_1\left(\frac{2\pi(i+1)\rho}{n},\frac{2\pi i\rho}{n}\right)\times\int_{\frac{i}{n}}^{\frac{i+1}{n}}g(s)\,ds\\ =\ &\frac{1}{\rho}\sum_{k=0}^{n-1}\left[(i+1)J_1\left(2\pi\frac{(i+1)}{n}\rho\right)-iJ_1\left(2\pi\frac{i}{n}\rho\right)\right]\times\int_{\frac{i}{n}}^{\frac{(i+1)}{n}}g(s)\,ds,\end{aligned} \tag{4.5}$$

then we can state the following corollary.

Corollary 5. *Let g be as in Theorem 3. Then we have*

$$G(g)(\rho) = H_n(g,\rho) + R_n(g,\rho), \ \rho \in [0,1], \tag{4.6}$$

where $H_n(g,\rho)$ is as given in (4.5) and the remainder $R_n(g,\rho)$ satisfies the estimate

$$|R_n(g,\rho)| \leq \frac{2\pi}{n} \bigvee_0^1 (g). \tag{4.7}$$

Remark 5. *Supposing that we know the total variation of g on the interval $[0,1]$ and would like to approximate $G(g)(\rho)$ with a given error $\varepsilon > 0$. Then we have to divided the interval $[0,1]$ into at least $n_\varepsilon \in \mathbb{N}^*$ points, where*

$$n_\varepsilon := \left[\frac{2\pi}{\varepsilon} \bigvee_0^1 (g) \right] + 1,$$

where $[x]$ is the integer part of $x \in \mathbb{R}$.

Remark 6. *Using the results and techniques outlined in papers* [1], [4]-[7]*, the authors intend to investigate the error estimates in approximating Hankel transforms of $r-$Hölder continuous, Lipschitzian or monotonic nondecreasing functions. The corresponding results will be comunicated in another paper that is in preparation.*

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