NOTE ON INTEGRAL VERSION OF THE GRÜSS INEQUALITY FOR COMPLEX FUNCTIONS

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Abstract. The corresponding version for complex-valued functions of a recent refinement of Grüss integral inequality due to Cerone & Dragomir is obtained.

1. Introduction

Let $(\Omega,\ \Sigma,\ \mu)$ be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L^2_\rho(\Omega,\mathbb{K})$ the Hilbert space of all measurable functions $f:\Omega\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\ \mathbb{R})$ that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(x)\left|f(x)\right|^2 d\mu(x)<0$, where $\rho:\Omega\to[0,\ \infty)$ is a given measurable function on Ω.

Dragomir and Gomm [3] have proved the following Grüss type result for complex valued functions:

Theorem 1. *Let $\varphi,\gamma,\Phi,\Gamma\in\mathbb{K}$ and $h,f,g\in L^2_\rho(\Omega,\mathbb{K})$ be such that*

$$\begin{aligned}\operatorname{Re}\left[(\Phi h(x)-f(x))\left(\bar{f}(x)-\bar{\varphi}\,\bar{h}(x)\right)\right] &\geq 0,\\ \operatorname{Re}\left[(\Gamma h(x)-g(x))\left(\bar{g}(x)-\bar{\gamma}\,\bar{h}(x)\right)\right] &\geq 0,\end{aligned}\tag{1.1}$$

for $\mu-$a.e. $x\in\Omega$ and $\int_\Omega\rho(x)\left|h(x)\right|^2 d\mu(x)=1$. Then we have inequality

$$\begin{aligned}&\left|\int_\Omega \rho(x)\,f(x)\ \bar{g}(x)\,d\mu(x)\right.\\ &\qquad\left.-\int_\Omega\rho(x)\,f(x)\,\bar{h}(x)\ d\mu(x)\int_\Omega\rho(x)\,\bar{g}(x)\,h(x)\,d\mu(x)\right|\\ &\qquad\qquad\leq\frac{1}{4}\left|\Phi-\varphi\right|\ \left|\Gamma-\gamma\right|.\end{aligned}\tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense that it cannot be replaced by a smaller quantity.

Denote by $T(f,g;\rho)$ the Čebyšev functional

$$\begin{aligned}&T(f,g;\rho)\\ :&=\frac{1}{R}\int_\Omega\rho(x)\,f(x)\bar{g}(x)\,d\mu(x)-\frac{1}{R}\int_\Omega\rho(x)\,f(x)d\mu(x)\,\frac{1}{R}\int_\Omega\rho(x)\bar{g}(x)\,d\mu(x),\end{aligned}$$

where, for simplicity, we denote $R:=\int_\Omega\rho(x)\,d\mu(x)>0$.

The following result was obtained in [3] as a simple consequence of Theorem 1.

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Corollary 1. *Let* $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ *and* $h, f, g \in L^2_\rho(\Omega,\mathbb{K})$ *be such that*

$$\begin{aligned} \operatorname{Re}\left[(\Phi - f(x))\left(\bar{f}(x) - \bar{\varphi}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(\Gamma - g(x))\left(\bar{g}(x) - \bar{\gamma}\right)\right] &\geq 0, \end{aligned} \tag{1.3}$$

for $\mu-$*a.e.* $x \in \Omega$. *Then we have the inequality*

$$|T(f, g; \rho)| \leq \frac{1}{4}|\Phi - \varphi|\,|\Gamma - \gamma|. \tag{1.4}$$

Here the constant $\frac{1}{4}$ *is also best possible.*

Let us note that we can prove the reverse implications as well, that is, Theorem 1 can be obtained as a consequence of Corollary 1.

Indeed, we have from the condition (1.1):

$$\begin{aligned} \operatorname{Re}\left[\frac{(\Phi h(x) - f(x))\left(\bar{f}(x) - \bar{\varphi}\,\bar{h}(x)\right)}{h(x)\,\bar{h}(x)}\right] &\geq 0, \\ \operatorname{Re}\left[\frac{(\Gamma h(x) - g(x))\left(\bar{g}(x) - \bar{\gamma}\,\bar{h}(x)\right)}{h(x)\,\bar{h}(x)}\right] &\geq 0, \end{aligned} \tag{1.5}$$

or

$$\begin{aligned} \operatorname{Re}\left[\left(\Phi - \frac{f(x)}{h(x)}\right)\left(\frac{\bar{f}(x)}{\bar{h}(x)} - \bar{\varphi}\right)\right] &\geq 0, \\ \operatorname{Re}\left[\left(\Gamma - \frac{g(x)}{h(x)}\right)\left(\frac{\bar{g}(x)}{\bar{h}(x)} - \bar{\gamma}\right)\right] &\geq 0. \end{aligned} \tag{1.6}$$

Using conditions (1.6) and inequality (1.4) in Corollary 1 with $\rho(x) := \rho(x)|h(x)|^2$, $f(x) := \frac{f(x)}{h(x)}$, $g(x) := \frac{g(x)}{h(x)}$ we get

$$\begin{aligned} &\left|\frac{1}{R}\int_\Omega \rho(x) f(x)\,\bar{g}(x)\,d\mu(x)\right. \\ &\qquad \left.- \frac{1}{R}\int_\Omega \rho(x) f(x)\,\bar{h}(x)\,d\mu(x)\frac{1}{R}\int_\Omega \rho(x)\,\bar{g}(x)\,h(x)\,d\mu(x)\right| \\ &\qquad\qquad \leq \frac{1}{4}|\Phi - \varphi|\,|\Gamma - \gamma|, \end{aligned} \tag{1.7}$$

i.e. the inequality (1.2) holds true.

For the real case, we also recall the following refinement of the classical Grüss inequality obtained in [1].

With the assumptions presented above and if

$$f \in L_\rho(\Omega, \mu) := \left\{f : \Omega \to \mathbb{R}, \int_\Omega \rho(x)|f(x)|\,d\mu(x) < 0\right\}$$

then we may define the functional

$$\begin{aligned} &D_\rho(f) \\ &:= \frac{1}{\int_\Omega \rho(x)\,d\mu(x)}\int_\Omega \rho(x)\left|f(x) - \frac{1}{\int_\Omega \rho(y)\,d\mu(y)}\int_\Omega \rho(y) f(y)\,d\mu(y)\right|d\mu(x). \end{aligned} \tag{1.8}$$

The following result holds [1].

Theorem 2. *Let $\rho, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $\rho \geq 0$ $\mu-$a.e. on Ω and $\int_\Omega \rho(y)\,d\mu(y) > 0$. If $f, g, fg \in L_\rho(\Omega, \mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for } \mu - a.e.\ x \in \Omega, \tag{1.9}$$

then we have the inequality

$$|T(f, g; \rho)| \leq \frac{1}{2}(\Delta - \delta) D_\rho(f). \tag{1.10}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

The above result naturally extend the one proved in Cheng and Sun [2] for functions of a real variable. The sharpness of the constant was not considered in [2]. The technique employed in [1] to prove the main results is different from the technique used by Cheng and Sun to obtain the inequality below:

Theorem 3. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\delta \leq g(x) \leq \Delta$ for some constants δ, Δ and for all $x \in [a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x) g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx\right| \leq \frac{\Delta - \delta}{2}\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,dy\right| dx. \tag{1.11}$$

The main aim of this note is to provide the corresponding refinement of Grüss inequality for complex-valued functions. Some related results are also pointed out.

2. The Results

The following preposition holds.

Proposition 1. *Let $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ and $f, g \in L^2_\rho(\Omega,\mathbb{K})$. Then we have inequality*

$$|T(f,\, g; \rho)| \leq \frac{1}{R}\int_\Omega \rho(x)\,|f(x) - \alpha| \left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\,\bar{g}(t)\,d\mu(t)\right| d\mu(x), \tag{2.1}$$

and every $\alpha \in \mathbb{K}$.

Proof. The following identity of Sonin type is valid:

$$T(f,\, g; \rho) = \frac{1}{R}\int_\Omega \rho(x)\,(f(x) - \alpha)\left(\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\,\bar{g}(t)\,d\mu(t)\right) d\mu(x). \tag{2.2}$$

Using well-known triangle inequality we get (2.1). ∎

Corollary 2. *Let $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ and $f, g \in L^2_\rho(\Omega,\mathbb{K})$. Then we have inequality*

$$|T(f,\, g;\, \rho)| \leq \frac{1}{R}\int_\Omega \rho(x)\left|f(x) - \frac{1}{R}\int_\Omega \rho(t)\,f(t)\,d\mu(t)\right| \times \left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\,\bar{g}(t)\,d\mu(t)\right| d\mu(x). \tag{2.3}$$

Proof. Applying inequality (2.2) in Theorem 1 with $\alpha = \frac{1}{R}\int_\Omega \rho(x)\,d\mu(x)$, we deduce the desired inequality (2.3). ∎

Remark 1. *Inequality (2.3) is a generalization of an inequality of S. S. Dragomir and McAndrew in* [4] *holding for functions of a real variable.*

The following *pre*-Grüss type inequality for complex-valued functions holds.

Theorem 4. *Let* φ, $\Phi \in \mathbb{K}$ *and* $f, g \in L^2_\rho(\Omega,\mathbb{K})$ *be such that*

$$Re\left[(\Phi - f(x))\left(\bar{f}(x) - \bar{\varphi}\right)\right] \geq 0, \tag{2.4}$$

for $\mu-$*a.e.* $x \in \Omega$, *or equivalently,*

$$\left|f(x) - \frac{\varphi + \Phi}{2}\right| \leq \frac{1}{2}|\Phi - \varphi|, \tag{2.5}$$

for $\mu-$*a.e.* $x \in \Omega$. *Then we have the inequalities*

$$\begin{aligned} &|T(f, g; \rho)| \\ &\leq \frac{1}{2}|\Phi - \varphi| \frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right| d\mu(x) \\ &\leq \frac{1}{2}|\Phi - \varphi| \left(\frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right|^2 d\mu(x)\right)^{\frac{1}{2}} \\ &= \frac{1}{2}|\Phi - \varphi| \left(\frac{1}{R}\int_\Omega \rho(x)|\bar{g}(x)|^2 d\mu(x) - \left|\frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

The constant $\frac{1}{2}$ *is best possible.*

Proof. The equivalence between (2.4) and (2.5) can be easily proved (see, for instance [5] where a proof in the general settings of inner product spaces is given). Using inequality (2.1) with $\alpha = \frac{\varphi+\Phi}{2}$ and (2.5), we have:

$$\begin{aligned} &|T(f, g; \rho)| \\ &\leq \frac{1}{R}\int_\Omega \rho(x)\left|f(x) - \frac{\varphi + \Phi}{2}\right|\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right| d\mu(x) \\ &\leq \frac{1}{2}|\Phi - \varphi| \frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right| d\mu(x). \end{aligned} \tag{2.7}$$

Now using the fundamental inequality for means ([6, p. 15]), we have

$$\begin{aligned} &\frac{1}{2}|\Phi - \varphi| \frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right| d\mu(x) \\ &\leq \frac{1}{2}|\Phi - \varphi| \left(\frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right|^2 d\mu(x)\right)^{\frac{1}{2}} \\ &= \frac{1}{2}|\Phi - \varphi| \left(\frac{1}{R}\int_\Omega \rho(x)|\bar{g}(x)|^2 d\mu(x) - \left|\frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.8}$$

The last identity in (2.8) is obvious.

The sharpness of the best constant follows from the real case, see for instance, [1], and we omit the details ∎

Corollary 3. *Let* $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ *and* $h, f, g \in L^2_\rho(\Omega, \mathbb{K})$ *be such that either*

$$\begin{aligned} Re\left[(\Phi - f(x))\left(\overline{f}(x) - \overline{\varphi}\right)\right] &\geq 0, \\ Re\left[(\Gamma - g(x))\left(\overline{g}(x) - \overline{\gamma}\right)\right] &\geq 0, \end{aligned}$$

for $\mu-$*a.e.* $x \in \Omega$, *or equivalently*

$$\left|f(x) - \frac{\varphi + \Phi}{2}\right| \leq \frac{1}{2}|\Phi - \varphi|, \tag{2.9}$$
$$\left|g(x) - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma|,$$

for $\mu-$*a.e.* $x \in \Omega$. *Then we have the inequalities*

$$\begin{aligned} &|T(f, g; \rho)| \\ &\leq \frac{1}{2}|\Phi - \varphi| \frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right| d\mu(x) \\ &\leq \frac{1}{2}|\Phi - \varphi| \left(\frac{1}{R}\int_\Omega \rho(x)\left|\bar{g}(x) - \frac{1}{R}\int_\Omega \rho(t)\bar{g}(t)\,d\mu(t)\right|^2 d\mu(x)\right)^{\frac{1}{2}} \\ &= \frac{1}{2}|\Phi - \varphi| \left(\frac{1}{R}\int_\Omega \rho(x)|g(x)|^2 d\mu(x) - \left|\frac{1}{R}\int_\Omega \rho(t)\overline{g}(t)\,d\mu(t)\right|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4}|\Phi - \varphi|\,|\Gamma - \gamma| \end{aligned} \tag{2.10}$$

The constant $\frac{1}{2}$ *is best possible.*

Proof. Using (2.6) we have

$$\begin{aligned} |T(g, g; \rho)|^2 &\leq \left|\frac{1}{R}\int_\Omega \rho(x)|g(x)|^2 d\mu(x) - \left|\frac{1}{R}\int_\Omega \rho(t)\overline{g}(t)\,d\mu(t)\right|^2\right|^2 \\ &\leq \frac{1}{4}|\Gamma - \gamma|^2 |T(g, g; \rho)|. \end{aligned} \tag{2.11}$$

Using (2.11) and (2.9), we may state that:

$$\begin{aligned} |T(g, g; \rho)| &\leq \frac{1}{R}\int_\Omega \rho(x)|g(x)|^2 d\mu(x) - \left|\frac{1}{R}\int_\Omega \rho(t)\overline{g}(t)\,d\mu(t)\right|^2 \\ &\leq \frac{1}{4}|\Gamma - \gamma|^2. \end{aligned} \tag{2.12}$$

Now, the inequality (2.12) and inequality (2.8) give the desired inequality (2.10). ∎

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