ERROR ESTIMATES FOR APPROXIMATING THE FOURIER TRANSFORM OF FUNCTIONS OF BOUNDED VARIATION

N.S. BARNETT, S.S. DRAGOMIR, AND G. HANNA

Abstract. In this paper we point out an approximation for the Fourier transform for functions of bounded variation and study the approximation error of certain associated quadrature rules.

1. Introduction

The *Fourier Transform* has applications in a wide variety of fields in science and engineering [1, p. xi].

In this paper, by use of some integral identities and inequalities developed in [4](see also [5]), we point out some approximations of the Fourier transform in terms of the complex exponential mean $E(z,w)$ (see Section 2) and study the error of approximation for different classes of mappings of bounded variation defined on finite intervals.

Let $g:[a,b]\to\mathbb{R}$ be a Lebesgue integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its Fourier transform, i.e.,

$$\mathcal{F}(g)(t):=\int_a^b g(s)\,e^{-2\pi its}ds.$$

The inverse Fourier transform of g will also be considered, and will be defined by

$$\mathcal{F}^{-1}(g)(t):=\int_a^b g(s)\,e^{2\pi its}ds.$$

The following inequality was obtained in [2]:

Theorem 1. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(t)\,dt\right|$$
$$\leq\begin{cases}\frac{1}{3}\|g'\|_\infty(b-a)^2, & \text{if } g'\in L_\infty[a,b],\\[2mm] \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p, & \text{if } g'\in L_p[a,b];\ \frac{1}{p}+\frac{1}{q}=1, p>1,\\[2mm] (b-a)\|g'\|_1\end{cases};$$

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for all $x \in [a,b]$, $x \neq 0$, where E is the exponential mean of two complex numbers, that is,

$$E(z,w) := \begin{cases} \frac{e^z - e^w}{z-w} & \text{if } z \neq w \\ \\ \exp(w) & \text{if } z = w \end{cases}, \quad z, w \in \mathbb{C}.$$

The following inequality for a more general class of functions was pointed out in [3].

Theorem 2. *Let $g : [a,b] \to \mathbb{R}$ be a measurable mapping on $[a,b]$, then we have the inequality:*

$$\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\,ds \right|$$
$$\leq \begin{cases} \frac{2\pi}{3}|x|(b-a)^2 \|g\|_\infty & \text{if } g \in L_\infty[a,b]; \\ \\ \frac{2^{1+\frac{1}{q}}\pi(b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}|x|\|g\|_p & \text{if } g \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ 2\pi|x|(b-a)\|g\|_1 & \text{if } g \in L_1[a,b]; \end{cases}$$

for all $x \in [a,b]$, $x \neq 0$.

It is the main aim of this paper to point out some new inequalities for the Fourier transform of functions of bounded variation. Error bounds for some associated quadrature formulae are also mentioned.

2. Some Inequalities

The following inequality holds:

Theorem 3. *Let $g : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$, then we have the inequality*

$$\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\,ds \right| \leq \frac{3}{4}(b-a)\bigvee_a^b(g) \tag{2.1}$$

for all $x \in [a,b]$, $x \neq 0$, where $\bigvee_a^b(g)$ is the total variation of g on $[a,b]$.

Proof. Using the integration by parts formula for the Riemann-Stieltjes integral, we have (see also [4]) that

$$\int_a^t (s-a)\,dg(s) = (t-a)g(t) - \int_a^t g(s)\,ds \tag{2.2}$$

and

$$\int_t^b (s-b)\,dg(s) = (b-t)g(t) - \int_t^b g(s)\,ds, \tag{2.3}$$

for all $t \in [a,b]$.
Adding (2.2) and (2.3) and dividing by $(b-a)$, we deduce the representation [4]:

$$g(t) = \frac{1}{b-a}\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^t (s-a)\,dg(s) + \frac{1}{b-a}\int_t^b (s-b)\,dg(s), \tag{2.4}$$

for all $t \in [a,b]$, which is itself of interest.
Assume that $x \in [a,b]$, $x \neq 0$, then,

$$
\begin{aligned}
\mathcal{F}(g)(x) &= \int_a^b g(t)\, e^{-2\pi i x t} dt \\
&= \int_a^b \left[\frac{1}{b-a} \int_a^b g(s)\, ds + \frac{1}{b-a} \int_a^t (s-a)\, dg(s) \right. \\
&\qquad \left. + \frac{1}{b-a} \int_t^b (s-b)\, dg(s) \right] e^{-2\pi i x t} dt \\
&= \frac{1}{b-a} \int_a^b g(s)\, ds \int_a^b e^{-2\pi i x t} dt \\
&\qquad + \frac{1}{b-a} \int_a^b \left(\int_a^t (s-a)\, dg(s) \right) e^{-2\pi i x t} dt \\
&\qquad + \frac{1}{b-a} \int_a^b \left(\int_t^b (s-b)\, dg(s) \right) e^{-2\pi i x t} dt \\
&= E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\, ds \\
&\qquad + \frac{1}{b-a} \int_a^b \left(\int_a^t (s-a)\, dg(s) \right) e^{-2\pi i x t} dt \\
&\qquad + \frac{1}{b-a} \int_a^b \left(\int_t^b (s-b)\, dg(s) \right) e^{-2\pi i x t} dt,
\end{aligned}
\tag{2.5}
$$

as

$$
\int_a^b e^{-2\pi i x t} dt = (b-a)\, E(-2\pi i x a, -2\pi i x b).
$$

Using the properties of modulus, we have, by (2.5), that

$$
\begin{aligned}
&\left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\, ds \right| \\
&\leq \frac{1}{b-a} \left| \int_a^b \left(\int_a^t (s-a)\, dg(s) \right) e^{-2\pi i x t} dt \right| \\
&\qquad + \frac{1}{b-a} \left| \int_a^b \left(\int_t^b (s-b)\, dg(s) \right) e^{-2\pi i x t} dt \right| \\
&\leq \frac{1}{b-a} \int_a^b \left| \int_a^t (s-a)\, dg(s) \right| \left| e^{-2\pi i x t} \right| dt \\
&\qquad + \frac{1}{b-a} \int_a^b \left| \int_t^b (s-b)\, dg(s) \right| \left| e^{-2\pi i x t} \right| dt \\
&= \frac{1}{b-a} \int_a^b \left| \int_a^t (s-a)\, dg(s) \right| dt + \frac{1}{b-a} \int_a^b \left| \int_t^b (s-b)\, dg(s) \right| dt.
\end{aligned}
\tag{2.6}
$$

It is well known that if $p:[c,d]\to\mathbb{R}$ is continuous and $v:[c,d]\to\mathbb{R}$ is of bounded variation on $[c,d]$, then the Riemann-Stieltjes integral $\int_c^d p(x)\,dv(x)$ exists and

$$\left|\int_c^d p(x)\,dv(x)\right| \le \sup_{x\in[c,d]}|p(x)|\bigvee_c^d(v). \tag{2.7}$$

Applying (2.7) on the intervals $[a,t]$ and $[t,b]$, we deduce that

$$\left|\int_a^t (s-a)\,dg(s)\right| \le (t-a)\bigvee_a^t(g),$$
$$\left|\int_t^b (s-b)\,dg(s)\right| \le (b-t)\bigvee_t^b(g)$$

and further that,

$$\begin{aligned}\left|\int_a^t (s-a)\,dg(s)\right| + \left|\int_t^b (s-b)\,dg(s)\right| &\le (t-a)\bigvee_a^t(g)+(b-t)\bigvee_t^b(g)\\ &\le \max_{t\in[a,b]}\{t-a,b-t\}\left[\bigvee_a^t(g)+\bigvee_t^b(g)\right]\\ &= \bigvee_a^b(g)\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right].\end{aligned}$$

Using (2.6),

$$\begin{aligned}&\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\\ &\le \frac{1}{b-a}\bigvee_a^b(g)\int_a^b\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]dt\\ &=\frac{3}{4}(b-a)\bigvee_a^b(g),\end{aligned}$$

as a simple calculation shows that

$$\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{(b-a)^2}{4},$$

and the theorem is completely proved. ∎

Remark 1. *If we consider the inverse Fourier transform*

$$\mathcal{F}^{-1}(g)(x)=\int_a^b g(t)\,e^{2\pi ixt}dt,$$

then, by a similar argument, we can prove that

$$\begin{aligned}&\left|\mathcal{F}^{-1}(g)(x)-E(2\pi ixa,2\pi ixb)\int_a^b g(s)\,ds\right|\\ &\qquad\le\frac{3}{4}(b-a)\bigvee_a^b(g),\ x\in[a,b],\ x\ne 0.\end{aligned} \tag{2.8}$$

The following corollaries are a natural consequence.

Corollary 1. *Let $g:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\leq\frac{3}{4}(b-a)\left|g(b)-g(a)\right|, \tag{2.9}$$

for all $x\in[a,b]$, $x\neq 0$.

The proof is obvious by Theorem 3, taking into account that every monotonic mapping is of bounded variation and $\bigvee_a^b(g)=|g(b)-g(a)|$.

Corollary 2. *Let $g:[a,b]\to\mathbb{R}$ be an $L-$Lipschitzian mapping on $[a,b]$, i.e.,*

$$|g(t)-g(s)|\leq L|t-s| \text{ for all } t,s\in[a,b]. \tag{L}$$

Then we have the inequality

$$\left|\mathcal{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\leq\frac{3}{4}L(b-a)^2. \tag{2.10}$$

The proof is obvious by Theorem 3, taking into account that if $g:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian, then L is of bounded variation on $[a,b]$ and $\bigvee_a^b(g)\leq L(b-a)$.

3. A Numerical Quadrature Formula

Let $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$, put $h_k:=x_{k+1}-x_k$ $(k=0,...,n-1)$ and $\nu(h):=\max\{h_k|k=0,...,n-1\}$. Define the sum (see also [2] and [3])

$$\mathcal{E}(g,I_n,x):=\sum_{k=0}^{n-1}E(-2\pi ixx_k,-2\pi ixx_{k+1})\times\int_{x_k}^{x_{k+1}}g(t)\,dt, \tag{3.1}$$

where $x\in[a,b]$, $x\neq 0$.

The following approximation theorem holds.

Theorem 4. *Let $g:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the quadrature rule*

$$\mathcal{F}(g)(x)=\mathcal{E}(g,I_n,x)+R(g,I_n,x); \tag{3.2}$$

where $\mathcal{E}(g,I_n,x)$ is as defined in (3.1) and the remainder $R(g,I_n,x)$ satisfies the estimate

$$|R(g,I_n,x)|\leq\frac{3}{4}\nu(h)\bigvee_a^b(g). \tag{3.3}$$

Proof. Applying Theorem 3 on every subinterval $[x_k,x_{k+1}]$, we can state that

$$\left|\int_{x_k}^{x_{k+1}}g(t)e^{-2\pi ixt}dt-E(-2\pi ixx_k,-2\pi ixx_{k+1})\times\int_{x_k}^{x_{k+1}}g(t)\,dt\right| \leq\frac{3}{4}h_k\bigvee_{x_k}^{x_{k+1}}(g),$$

for all $k \in \{0, ..., n-1\}$ and $x \in [a, b]$, $x \neq 0$.
Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we can state that

$$\begin{aligned}|R(g, I_n, x)| &= |\mathcal{F}(g)(x) - \mathcal{E}(g, I_n, x)| \\ &\leq \frac{3}{4} \sum_{k=0}^{n-1} h_k \bigvee_{x_k}^{x_{k+1}} (g) \leq \frac{3}{4} \nu(h) \sum_{k=0}^{n-1} \bigvee_{x_k}^{x_{k+1}} (g) \\ &= \frac{3}{4} \nu(h) \bigvee_a^b (g),\end{aligned}$$

and the theorem is proved. ∎

In practical applications, it is more convenient to consider the equidistant partitioning of the interval $[a, b]$. Thus, let

$$I_n : x_j = a + j \cdot \frac{b-a}{n},\ j = 0, ..., n;$$

be an equidistant partition of $[a, b]$, and define the sum (see also [2] and [3])

$$\mathcal{E}_n(g, x) := \sum_{k=0}^{n-1} E\left[-2\pi i x \left(a + k \cdot \frac{b-a}{n}\right), -2\pi i x \left(a + (k+1) \cdot \frac{b-a}{n}\right)\right] \times \int_{a+k \cdot \frac{b-a}{n}}^{a+(k+1) \cdot \frac{b-a}{n}} g(t)\, dt. \tag{3.4}$$

The following corollary of Theorem 4 holds.

Corollary 3. *Let g be as defined in Theorem 4. Then we have*

$$\mathcal{F}(g)(x) = \mathcal{E}_n(g, x) + R_n(g, x), \tag{3.5}$$

where $\mathcal{E}_n(g, x)$ approximates the Fourier transform at any point $x \in [a, b]$, $x \neq 0$. The error of approximation $R_n(g, x)$ satisfies the bound

$$|R_n(g, x)| \leq \frac{3}{4n}(b-a) \bigvee_a^b (g), \tag{3.6}$$

for all $x \in [a, b]$, $x \neq 0$.

Remark 2. *If we know the total variation $\bigvee_a^b (g)$ of g on $[a, b]$ and would like to approximate the Fourier transform $F(g)(x)$ by the adaptive quadrature formula $\mathcal{E}_n(g, x)$ with an error less than a given $\varepsilon > 0$, we have to divide the interval $[a, b]$ into at least $n_\varepsilon \in \mathbb{N}$ points, where*

$$n_\varepsilon := \left[\frac{3(b-a)}{4\varepsilon} \bigvee_a^b (g)\right] + 1,$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

The following corollaries of Theorem 4 also hold.

Corollary 4. *Let $g : [a, b] \to \mathbb{R}$ be a monotonic mapping on $[a, b]$. Then we have the quadrature formula* (3.2) *where the remainder is such that it satisfies the estimate*

$$|R(g, I_n, x)| \leq \frac{3}{4}\nu(h)|g(b) - g(a)|, \ x \in [a, b], \ x \neq 0. \tag{3.7}$$

In particular, if I_n is taken to be equidistant, then we have the formula (3.5), *where the remainder $R_n(g, x)$ satisfies the estimate*

$$|R_n(g, x)| \leq \frac{3(b-a)}{4n}|g(b) - g(a)|, \ x \in [a, b], \ x \neq 0. \tag{3.8}$$

A similar result holds for Lipschitzian mappings.

Corollary 5. *Let $g : [a, b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the quadrature formula* (3.2) *where the remainder satisfies the bound*

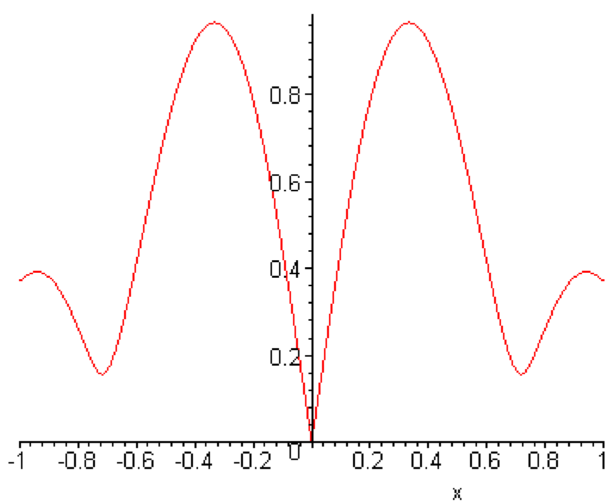
$$|R(g, I_n, x)| \leq \frac{3}{4}L\sum_{i=0}^{n-1} h_i^2 \leq \frac{3}{4}L(b-a)\nu(h). \tag{3.9}$$

In particular, if I_n is chosen to be equidistant, then we have the formula (3.5) *where the remainder $R_n(g, x)$ is such that it satisfies the inequality*

$$|R_n(g, x)| \leq \frac{3L(b-a)^2}{4n}. \tag{3.10}$$

4. Some Numerical Experiments

In the following we numerically illustrate the approximation for the Fourier transform provided by



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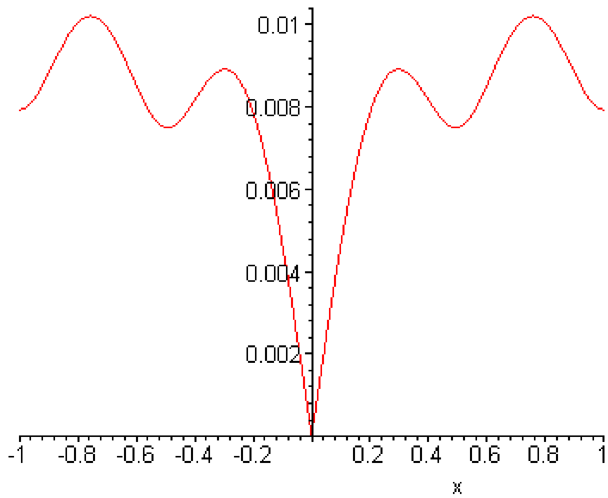

,

$$\begin{aligned} (4.1) \qquad & \mathcal{E}(g, I_n, x) \\ : & = \sum_{k=0}^{n-1} E\left(2\pi i x\left(a + k\cdot\frac{b-a}{n}\right), -2\pi\, i\left(a + (k+1)\frac{b-a}{n}\right)\right) \\ & \times \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\frac{b-a}{n}} g(t)\,dt. \end{aligned}$$

If we consider, for instance, the exponential function $g(t) = \exp(t), t \in [-1, 1]$, then the plots of the error $r_n(x) := |R_n(g, x)|, x \in [-1, 1]$ for $n = 1, n = 10$ and $n = 100$, respectively, are depicted in Figure 1, 2 and 3.

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