

# SPECTRAL RADII OF OPERATORS AND HIGH-POWER OPERATOR INEQUALITIES

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ABSTRACT. For some different types of operators on a Hilbert space, we present new high-power operator inequalities, and their corresponding operator inequalities involving spectral radii of operators. We prove that each such operator inequality is equivalent to the Cauchy-Schwarz inequality. In particular, we show that Halmos' two operator inequalities, Reid's inequality, and many others hold easily. We obtain a new generalized Löwner inequality, and a short proof of the classical Löwner-Heinz inequality is given.

## 1. INTRODUCTION

The Cauchy-Schwarz inequality is a powerful inequality which states that the ratio

$$(1.1) \quad |(x, y)| \leq \|x\| \|y\|$$

holds for every  $x$  and  $y$  in a pre-Hilbert space. Every inequality in this space is either derived from the Cauchy-Schwarz inequality, or equivalent to it. For recent developments on inequalities related to (1.1) see [1] and the references therein.

In this paper we use capital letters to denote bounded linear operators on a complex Hilbert space  $H$ , and  $I$  denotes the identity operator. A positive operator  $T$  is written as  $T \geq 0$ , the zero operator. We shall consider four types of high-power operator inequalities, and their corresponding operator inequalities involving spectral radii of operators. Four types are: a positive operator, two arbitrary operators, mixed operators, and two selfadjoint operators. Indeed, our results are motivated by Halmos' two operator inequalities in [2, p. 51 and 244]. He proved that if  $T \geq 0$ ,  $S$  is arbitrary and  $TS$  is selfadjoint operators, then, for every  $x \in H$ , the following high-power operator inequality holds

$$(1.2) \quad |(TSx, x)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^n-1}$$

for  $n \geq 0$ . From this he concluded that the inequality involving spectral radius

$$(1.3) \quad |(TSx, x)| \leq r(S)(Tx, x)$$

holds, where  $r(S)$  means the spectral radius of  $S$ . It is a stronger version of a result due to Reid [6]; Reid had  $\|S\|$  instead of  $r(S)$  (that  $r(S) \leq \|S\|$  is known [2, p. 45]). Actually, we prove that some generalizations of inequalities (1.2), (1.3), Reid's inequality, and many others are all equivalent to the Cauchy-Schwarz inequality. In particular, it is shown that inequalities (1.2), (1.3), Reid's inequality, and many

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others hold easily. We also obtain a new generalized Löwner inequality, and a proof that the Cauchy-Schwarz inequality implies the classical Löwner-Heinz inequality, which is essential in operator inequalities on  $H$ . Finally we pose a question.

## 2. RESULTS

First of all, recall that the inequality  $|(Tx, y)|^2 \leq (Tx, x)(Ty, y)$  holds for  $T \geq 0$  and for all  $x, y \in H$  (consider the unique positive square root of  $T$ ). In fact, it is known to be equivalent to the Cauchy-Schwarz inequality, and this is crucial in the proof of our results. Next, we need a well known relation:  $r(S) = \lim_n \|S^n\|^{1/n}$  for any operator  $S$  [2, Problem 74].

**Theorem 1.** *Let  $T \geq 0$ , and  $S$  and  $C$  be arbitrary operators. Also let  $TS, TC, A$  and  $B$  be all selfadjoint operators. If  $n$  is a positive integer, then for all  $x, y \in H$ ,  $y \neq x$ , the following are equivalent to one another and to the Cauchy-Schwarz inequality (1.1):*

$$(2.1) \quad |(Tx, y)|^{2^n} \leq (T^{1+2^{n-1}}x, x)(Tx, x)^{2^{n-1}-1} \|y\|^{2^n} \quad \text{for } n \geq 1; \quad \text{and}$$

$$(2.2) \quad |(Tx, y)|^2 \leq r(T)(Tx, x) \|y\|^2;$$

$$(2.3) \quad |(Sx, Cy)|^{2^n} \leq ((S^*S)^{2^{n-1}}x, x)((C^*C)^{2^{n-1}}y, y) \|x\|^{2^{n-2}} \|y\|^{2^{n-2}} \\ \text{for } n \geq 1; \quad \text{and}$$

$$(2.4) \quad |(Sx, Cy)|^2 \leq r(S^*S)r(C^*C) \|x\|^2 \|y\|^2;$$

$$(2.5) \quad |(TSx, Cy)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^{n-1}-1}(TC^{2^n}y, y)(Ty, y)^{2^{n-1}-1} \\ \text{for } n \geq 1; \quad \text{and}$$

$$(2.6) \quad |(TSx, Cy)| \leq r(S)r(C)(Tx, x)^{\frac{1}{2}}(Ty, y)^{\frac{1}{2}};$$

$$(2.7) \quad |(Ax, By)|^{2^n} \leq (A^{2^{n-1}+2}x, x)(B^{2^{n-1}+2}y, y) \|Ax\|^{2^{n-1}-2} \|By\|^{2^{n-1}-2} \\ \times \|x\|^{2^{n-1}} \|y\|^{2^{n-1}-1} \quad \text{for } n \geq 2; \quad \text{and}$$

$$(2.8) \quad |(Ax, By)|^2 \leq r(A)r(B) \|Ax\| \|By\| \|x\| \|y\|.$$

*Proof.* It is trivial to show that any one of statements (2.2), (2.4), (2.6) and (2.8) implies (1.1); just letting  $T = S = C = A = B = I$  will suffice.

**(1.1)  $\Rightarrow$  (2.1).** We shall prove it inductively as follows, and start with  $n = 1$  first.

$$|(Tx, y)|^2 \leq (T^2x, x) \|y\|^2.$$

As

$$(T^2x, x)^2 \leq (TTx, Tx)(Tx, x) = (T^3x, x)(Tx, x),$$

we have

$$|(Tx, y)|^4 \leq (T^3x, x)(Tx, x) \|y\|^4$$

for  $n = 2$ . Since

$$(T^{1+2^{n-1}}x, x)^2 \leq (TT^{2^{n-1}}x, T^{2^{n-1}}x)(Tx, x) = (T^{1+2^n}x, x)(Tx, x),$$

we obtain

$$\begin{aligned} |(Tx, y)|^{2^{n+1}} &\leq \left[ |(Tx, y)|^{2^n} \right]^2 \\ &\leq \left[ (T^{1+2^{n-1}}x, x)(Tx, x)^{2^{n-1}-1} \|y\|^{2^n} \right]^2 \\ &\leq (T^{1+2^n}x, x)(Tx, x)^{2^n-1} \|y\|^{2^{n+1}}, \end{aligned}$$

and the induction process is completed.

**(2.1)**  $\Rightarrow$  **(2.2)**. The inequality (2.1) gives

$$|(Tx, y)|^{2^n} \leq \|T\| \left\| T^{2^{n-1}} \right\| \|x\|^2 (Tx, x)^{2^{n-1}-1} \|y\|^{2^n}.$$

Taking the  $2^{n-1}$ -th root of both sides of the inequality above yields

$$|(Tx, y)|^2 \leq \|T\|^{\frac{1}{2^{n-1}}} \left\| T^{2^{n-1}} \right\|^{\frac{1}{2^{n-1}}} \|x\|^{\frac{2}{2^{n-1}}} (Tx, x)^{1-\frac{1}{2^{n-1}}} \|y\|^2.$$

Passing to the limit as  $n \rightarrow \infty$  we have the desired conclusion.

We mention before we continue that the methods of the proof of all others are similar to above.

**(1.1)**  $\Rightarrow$  **(2.3)**.  $|(Sx, Cy)|^2 \leq (S^*Sx, x)(C^*Cy, y)$  for  $n = 1$ .

For the inductive step, note first that

$$((S^*S)^{2^{n-1}}x, x)^2 \leq ((S^*S)^{2^n}x, x) \|x\|^2.$$

So,

$$\begin{aligned} |(Sx, Cy)|^{2^{n+1}} &\leq \left[ \left( (S^*S)^{2^{n-1}}x, x \right) \left( (C^*C)^{2^{n-1}}y, y \right) \|x\|^{2^n-2} \|y\|^{2^n-2} \right]^2 \\ &= \left( (S^*S)^{2^n}x, x \right) \left( (C^*C)^{2^n}y, y \right) \|x\|^{2^{n+1}-2} \|y\|^{2^{n+1}-2}, \end{aligned}$$

and the process is completed.

**(2.3)**  $\Rightarrow$  **(2.4)**. The inequality (2.3) gives

$$|(Sx, Cy)|^{2^n} \leq \left\| (S^*S)^{2^{n-1}} \right\| \left\| (C^*C)^{2^{n-1}} \right\| \|x\|^{2^n} \|y\|^{2^n},$$

which yields

$$|(Sx, Cy)|^2 \leq \left\| (S^*S)^{2^{n-1}} \right\|^{\frac{1}{2^{n-1}}} \left\| (C^*C)^{2^{n-1}} \right\|^{\frac{1}{2^{n-1}}} \|x\|^2 \|y\|^2.$$

(2.4) follows immediately if we take the limit in above as  $n \rightarrow \infty$ .

**(1.1)**  $\Rightarrow$  **(2.5)**. As  $T$  is positive and both  $TS$  and  $TC$  are selfadjoint, we see that  $S^*TS = (TS)^*S = TS^2$ . And by induction we get  $(S^*)^i TS^i = TS^{2i}$  (for  $i = 1, 2, \dots$ ). Similarly,  $(C^*)^i TC^i = TC^{2i}$  (for  $i = 1, 2, \dots$ ). It follows, for  $n = 1$ , that

$$|(TSx, Cy)|^2 \leq (TSx, Sx)(TCy, Cy) = (TS^2x, x)(TC^2y, y).$$

Since

$$(TS^{2^n}x, x)^2 \leq ((S^*)^{2^n}TS^{2^n}x, x)(Tx, x) = (TS^{2^{n+1}}x, x)(Tx, x),$$

we have

$$\begin{aligned} |(TSx, Cy)|^{2^{n+1}} &\leq (TS^{2^n}x, x)^2 (Tx, x)^{2^n-2} (TC^{2^n}y, y)^2 (Ty, y)^{2^n-2} \\ &\leq (TS^{2^{n+1}}x, x) (Tx, x)^{2^n-1} (TC^{2^{n+1}}y, y) (Ty, y)^{2^n-1}. \end{aligned}$$

This proves, by induction, the inequality (2.5).

**(2.5)  $\Rightarrow$  (2.6).** (2.5) yields

$$|(TSx, Cy)|^{2^n} \leq \|T\|^2 \left\| S^{2^n} \right\| \left\| C^{2^n} \right\| \|x\|^2 \|y\|^2 (Tx, x)^{2^{n-1}-1} (Ty, y)^{2^{n-1}-1},$$

which implies, by taking the  $2^n$ -th root,

$$\begin{aligned} |(TSx, Cy)| &\leq \|T\|^{\frac{1}{2^{n-1}}} \left\| S^{2^n} \right\|^{\frac{1}{2^n}} \left\| C^{2^n} \right\|^{\frac{1}{2^n}} \\ &\quad \times \|x\|^{\frac{1}{2^{n-1}}} \|y\|^{\frac{1}{2^{n-1}}} (Tx, x)^{\frac{1}{2} - \frac{1}{2^n}} (Ty, y)^{\frac{1}{2} - \frac{1}{2^n}}. \end{aligned}$$

Thus, we have the inequality (2.6) after passing to the limit as  $n \rightarrow \infty$ .

**(1.1)  $\Rightarrow$  (2.7).** Since  $|(Ax, By)|^2 \leq (A^2x, x)(B^2y, y)$ ,

$$\begin{aligned} |(Ax, By)|^4 &\leq (A^2x, x)^2 (B^2y, y)^2 \\ &\leq (A^2x, A^2x)(B^2y, B^2y) \|x\|^2 \|y\|^2 \\ &= (A^4x, x)(B^4y, y) \|x\|^2 \|y\|^2 \end{aligned}$$

for  $n = 2$ . Note that  $A^2 \geq 0$ , and

$$(A^{2^{n-1}+2}x, x)^2 = (A^2A^{2^{n-1}}x, x)^2 \leq (A^{2^n+2}x, x) \|Ax\|^2,$$

and similarly for  $B^2 \geq 0$ . Therefore,

$$|(Ax, By)|^{2^{n+1}} \leq (A^{2^n+2}x, x)(B^{2^n+2}y, y) \|Ax\|^{2^n-2} \|By\|^{2^n-2} \|x\|^{2^n} \|y\|^{2^n},$$

and (2.7) holds by induction.

**(2.7)  $\Rightarrow$  (2.8).** The inequality (2.7) gives

$$\begin{aligned} |(Ax, By)|^{2^n} &\leq \|A\|^2 \left\| A^{2^{n-1}} \right\| \|Ax\|^{2^{n-1}-2} \\ &\quad \times \|B\|^2 \left\| B^{2^{n-1}} \right\| \|By\|^{2^{n-1}-2} \|x\|^{2^{n-1}+2} \|y\|^{2^{n-1}+2}. \end{aligned}$$

The next step is taking the  $2^{n-1}$ -th root, and then passing to the limit as  $n \rightarrow \infty$ ; the same as we did many times before. The proof of the theorem is now completed. ■

By a well-known result that if  $E$  is a normal operator (selfadjoint operator, in particular) on a complex Hilbert space, then  $r(E) = \|E\|$  [7, Theorem 6.2-E]. Thus, the proofs of (1.1) $\Leftrightarrow$ (2.2), (1.1) $\Leftrightarrow$ (2.4) and (1.1) $\Leftrightarrow$ (2.8) in Theorem 1 are trivial. However, our proofs do not rely on this result. It should be pointed out that (2.5) and (2.6) in Theorem 1 are generalizations of Halmos' inequalities (1.2) and (1.3), respectively. The next result, a generalization of Reid's inequality, is obviously a consequence of (2.6) in Theorem 1 and the proof should be omitted.

**Corollary 1.** *Let  $T \geq 0$ , and  $S$  and  $C$  be arbitrary operators. If  $TS$  and  $TC$  are selfadjoint operators, then for all  $x, y \in H$ ,  $y \neq x$ , the following inequality is equivalent to (1.1):*

$$(2.9) \quad |(TSx, Cy)| \leq \|S\| \|C\| (Tx, x)^{\frac{1}{2}} (Ty, y)^{\frac{1}{2}}.$$

The Cauchy-Schwarz inequality (1.1) can produce various kinds of inequalities which are not immediately apparent. The next results are consequences of Theorem 1 and Corollary 1. This also shows why the condition  $y \neq x$  is imposed in both results.

**Corollary 2.** *Let  $T \geq 0$ , and  $S$  and  $C$  be arbitrary operators. Also let  $TS$ ,  $TC$ ,  $A$  and  $B$  be all selfadjoint operators. If  $n$  is a positive integer, then for every  $x \in H$  the following hold:*

$$(2.10) \quad (Tx, x)^{2^{n-1}+1} \leq (T^{1+2^{n-1}}x, x) \|x\|^{2^n} \quad \text{for } n \geq 1;$$

$$(2.11) \quad (Tx, x) \leq r(T) \|x\|^2;$$

$$(2.12) \quad |(Sx, Cx)|^{2^n} \leq ((S^*S)^{2^{n-1}}x, x)((C^*C)^{2^{n-1}}x, x) \|x\|^{2^{n+1}-4} \quad \text{for } n \geq 1;$$

$$(2.13) \quad |(Sx, x)|^{2^n} \leq ((S^*S)^{2^{n-1}}x, x) \|x\|^{2^{n+1}-2} \quad \text{for } n \geq 1;$$

$$(2.14) \quad |(Sx, Cx)|^2 \leq r(S^*S)r(C^*C) \|x\|^4;$$

$$(2.15) \quad |(Sx, x)|^2 \leq r(S^*S) \|x\|^4;$$

$$(2.16) \quad |(TSx, Cx)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^n-2}(TC^{2^n}x, x) \quad \text{for } n \geq 1;$$

$$(2.17) \quad |(TSx, x)|^{2^n} \leq (TS^{2^n}x, x)(Tx, x)^{2^n-1} \quad \text{for } n \geq 0$$

*(Halmos' inequality (1.2));*

$$(2.18) \quad |(TSx, Cx)| \leq r(S)r(C)(Tx, x);$$

$$(2.19) \quad |(TSx, x)| \leq r(S)(Tx, x) \quad \text{(Halmos' inequality (1.3))};$$

$$(2.20) \quad |(Ax, Bx)|^{2^n} \leq (A^{2^{n-1}+2}x, x)(B^{2^{n-1}+2}x, x) \\ \times \|Ax\|^{2^{n-1}-2} \|Bx\|^{2^{n-1}-2} \|x\|^{2^{n+1}-2} \quad \text{for } n \geq 2;$$

$$(2.21) \quad |(Ax, x)|^{2^n} \leq (A^{2^{n-1}+2}x, x) \|Ax\|^{2^{n-1}-2} \|x\|^{5(2^{n-1})-2} \quad \text{for } n \geq 2;$$

$$(2.22) \quad |(Ax, Bx)|^2 \leq r(A)r(B) \|Ax\| \|Bx\| \|x\|^2.$$

$$(2.23) \quad |(Ax, x)|^2 \leq r(A) \|Ax\| \|x\|^3;$$

$$(2.24) \quad |(TSx, Cx)| \leq \|S\| \|C\| (Tx, x);$$

$$(2.25) \quad |(TSx, x)| \leq \|S\| (Tx, x) \quad \text{(Reid's inequality)}.$$

*Proof.* The proof is simple. Let, in particular,  $y = x$  in Theorem 1 and Corollary 1 above, so that the Cauchy-Schwarz inequality, (1.1) and (2.10) in Corollary 2, becomes the trivial case  $(x, x) = \|x\|^2$ . ■

The classical Löwner-Heinz inequality was initiated in [4] and established in [5], which is a basic tool in theory of operator inequalities on  $H$ . More precisely, the inequality  $P^\alpha \geq Q^\alpha$  holds if  $P \geq Q \geq 0$ , where  $\alpha \in [0, 1]$ . There are known examples showing that the inequality does not hold in general if  $\alpha > 1$ . The proof of the inequality was neither elementary nor short. However, there is a classical characterization of the inequality, namely  $P^{\frac{1}{2}} \geq Q^{\frac{1}{2}}$  holds if  $P \geq Q \geq 0$ , which is known as the Löwner inequality. We propose next a new proof that the Löwner-Heinz inequality may follow by way of the Cauchy-Schwarz inequality (Corollary 3 below). First of all, more generally we have

**Theorem 2.** *The Cauchy-Schwarz inequality implies a generalized Löwner inequality, i.e.,*

$$r(C)P^{\frac{1}{2}} \geq C^*Q^{\frac{1}{2}}$$

if  $P \geq Q \geq 0$ , both  $P^{\frac{1}{2}}C$  and  $C^*Q^{\frac{1}{2}}$  are selfadjoint for some operator  $C$ .

*Proof.* It suffices to show that a slightly generalized Reid's inequality (2.18) in Corollary 2 implies the required inequality. Now, we may assume without loss of generality that  $P$  is invertible, then  $P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \leq I$  as  $P \geq Q \geq 0$ . Let  $S = P^{-\frac{1}{2}}Q^{\frac{1}{2}}$ . Then  $SS^* = P^{-\frac{1}{2}}QP^{-\frac{1}{2}} \leq I$ , i.e.,  $S$  is a contraction. Next, let  $T = P^{\frac{1}{2}} \geq 0$ , then  $C^*TS = C^*Q^{\frac{1}{2}}$ . As both  $P^{\frac{1}{2}}C$  and  $C^*Q^{\frac{1}{2}}$  are selfadjoint by assumption (thus,  $T \geq 0$ , and both  $TS$  and  $TC$  are selfadjoint), it follows from the inequality  $|(TSx, Cx)| \leq r(S)r(C)(Tx, x)$  that

$$\left( C^*Q^{\frac{1}{2}}x, x \right) \leq r(S) \left( r(C)P^{\frac{1}{2}}x, x \right) \leq \left( r(C)P^{\frac{1}{2}}x, x \right)$$

for every  $x \in H$ . ■

**Corollary 3.** *The Cauchy-Schwarz inequality implies the Löwner-Heinz inequality.*

*Proof.* It suffices to show that (2.19) (Halmos' inequality (1.3)) in Corollary 2 implies the Löwner inequality. This is precisely the inequality in Theorem 2, where we let  $C = I$ . ■

As usual, let  $|E|$  mean the positive square root of the positive operator  $E^*E$ .

**Corollary 4.** *Let  $T \geq 0$  and  $TS$  be a selfadjoint operator. Then the following are equivalent.*

- (1)  $|(TS|x, x)| \leq \|S\| (Tx, x)$  for every  $x \in H$ ;
- (2)  $|(TSx, x)| \leq \|S\| (Tx, x)$  for every  $x \in H$  (Reid's inequality);
- (3)  $P^{\frac{1}{2}} \geq Q^{\frac{1}{2}}$  if  $P \geq Q \geq 0$  (Löwner inequality).

*Proof.*

(1) $\Rightarrow$ (2). We use a familiar relation that  $-|A| \leq A \leq |A|$  holds if  $A$  is selfadjoint. In other words,  $|(Ax, x)| \leq (|A|x, x)$  for every  $x \in H$ .

(2) $\Rightarrow$ (3). In the proof of Theorem 2 let  $C = I$  and use (2.25) in Corollary 2 instead of (2.18).

(3) $\Rightarrow$ (1). Since  $S/\|S\|$  is a contraction, i.e.,  $SS^* \leq \|S\|^2 I$ , we have

$$0 \leq (TS)^2 = TS(TS)^* = TSS^*T \leq \|S\|^2 T^2.$$

It follows from (2.12) that  $|TS| \leq \|S\| T$ . Therefore,

$$|(TS|x, x)| \leq (\|S\| Tx, x) = \|S\| (Tx, x).$$

Notice that the equivalence of the Reid's inequality and the Löwner-Heinz inequality has been pointed out in [8]. In conclusion, in view of Corollary 4, let us pose a question:

**Problem:** *Could we prove that the generalized Löwner inequality in Theorem 2 implies the inequality (2.18) in Corollary 2? In other words, are the two inequalities equivalent?*

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