SOME REFINEMENTS OF KY FAN'S INEQUALITY

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Abstract. We give some refinements of Ky Fan's inequality and also prove some inequalities involving the symmetric means.

1. Introduction

Let $M_{n,r}(\mathbf{x})$ be the generalized weighted power means: $M_{n,r}(\mathbf{x}) = (\sum_{i=1}^n \omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^n \omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $M_{n,0}(\mathbf{x})$ denotes the limit of $M_{n,r}(\mathbf{x})$ as $r \to 0^+$. Unless specified, we always assume $0 < x_1 \leq x_2 \cdots \leq x_n$. We denote $\sigma_n = \sum_{i=1}^n \omega_i (x_i - A_n)^2$.

To any given $\mathbf{x}, t \geq 0$ we associate $\mathbf{x}' = (1 - x_1, 1 - x_2, \cdots, 1 - x_n), \mathbf{x}_t = (x_1 + t, \cdots, x_n + t)$. When there is no risk of confusion, we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x})$, $M_{n,r,t}$ for $M_{n,r}(\mathbf{x}_t)$ and $M'_{n,r}$ for $M_{n,r}(\mathbf{x}')$ if $x_n < 1$. The meaning of $P_s, P'_s, P_{s,t}$ are similar. We also define $A_n = M_{n,1}, G_n = M_{n,0}, H_n = M_{n,-1}$ and similarly for $A'_n, G'_n, H'_n, A_{n,t}, G_{n,t}, H_{n,t}$.

Recently, the author[7] proved the following result.

Theorem 1.1. *For $r > s, x_1 > 0$, the following inequalities are equivalent:*

$$\frac{r-s}{2x_1}\sigma_n \geq \quad M_{n,r} - M_{n,s} \quad \geq \frac{r-s}{2x_n}\sigma_n, \tag{1.1}$$

$$\frac{x_n}{1-x_n}(M_{n,r} - M_{n,s}) \geq \quad M'_{n,r} - M'_{n,s} \quad \geq \frac{x_1}{1-x_1}(M_{n,r} - M_{n,s}), \tag{1.2}$$

where in (1.2) *we require $x_n < 1$.*

For extensions and refinements of (1.1), see [2], [9],[12] and [13]. Inequality (1.2) is commonly referred as the additive Ky Fan's inequality. We refer the reader to the survey article[1] and the references therein for an account of Ky Fan's inequality.

D.Cartwright and M.Field[4] first proved the validity of (1.1) for $r = 1, s = 0$. Under the assumption $x_n \leq 1/2$, it is easy to show(see [6]) if $\beta \leq \alpha$, then $A_n^\alpha - G_n^\alpha \geq A_n^{'\alpha} - G_n^{'\alpha}$ implies $A_n^\beta - G_n^\beta \geq A_n^{'\beta} - G_n^{'\beta}$ and $A_n^\beta - G_n^\beta \leq A_n^{'\beta} - G_n^{'\beta}$ implies $A_n^\alpha - G_n^\alpha \leq A_n^{'\alpha} - G_n^{'\alpha}$. Thus if $x_n \leq 1/2$, the above Theorem then implies $A_n^\alpha - G_n^\alpha \geq A_n^{'\alpha} - G_n^{'\alpha}$ for $\alpha \leq 1$. Alzer[3] has given a counter example to show that $A_n^\alpha - G_n^\alpha$ and $A_n^{'\alpha} - G_n^{'\alpha}$ are not comparable in general for any fixed $\alpha > 1$. It is then interesting to seek for certain $\alpha > 1$, as a function of the weights so that $A_n^\alpha - G_n^\alpha$ and $A_n^{'\alpha} - G_n^{'\alpha}$ are comparable. One motivation is the following result of Pečarić and Alzer[15](see also [1], Theorem 7.2).

Theorem 1.2. *For $\omega_i = 1/n, 0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq 1/2$,*

$$A_n^n - G_n^n \leq A_n^{'n} - G_n^{'n}. \tag{1.3}$$

Theorem 1.2 suggests that $A_n^\alpha - G_n^\alpha \leq A_n^{'\alpha} - G_n^{'\alpha}$ for $\alpha = 1/q$ with $q = \min\{\omega_i\}$. We will show this is indeed the case in section 3. A similar result is also proved there. The idea of the proof of (1.3) also allows us to establish some inequalities involving the symmetric means in section 4.

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2. Lemmas

Lemma 2.1. *For $0 < q < 1$, $0 < G_n \le A_n \le 1, f(q) = 2q(A_n^{\frac{1}{q}} - G_n^{\frac{1}{q}})$ is an increasing function of q.*

Proof. Let $x = A_n, y = G_n$, then $f'(q) = 2(x^{\frac{1}{q}} - y^{\frac{1}{q}}) - 2(\ln(x^{\frac{1}{q}})x^{\frac{1}{q}} - \ln(y^{\frac{1}{q}})y^{\frac{1}{q}}) \ge 0$, since $u - u\ln u$ increases with respect to u for $0 < u \le 1$. □

Lemma 2.2. *For $0 < q \le 1$, $(1-q)^{1/q-1}$ is an increasing function of q, in particular, $(1-q)^{1/q-1} \le 1/2$ when $0 < q \le 1/2$ and the above inequality reverses when $1/2 \le q < 1$. In either case, equality holds if and only if $q = 1/2$.*

Proof. It suffices to show $f'(q) \ge 0$ for $0 < q < 1$ with $f(q) = (1/q - 1)\ln(1 - q)$. Now $f'(q) = -h(q)/q^2$ with $h(q) = q + \ln(1 - q) < 0$ for $0 < q < 1$, we are done. □

3. The Main Results

To motivate our next result, we note that L. Hoehn and I. Niven[10] showed $A_{n,t} - G_{n,t}$ is a decreasing function of t. It then follows that $f(t, \alpha) = A^{\alpha}_{n,t} - G^{\alpha}_{n,t}$ is decreasing as a function of t(See [8], Theorem 2.1) for $\alpha \le 1$. It's natural to ask whether one can have similar results for $\alpha \ge 1$ and we have the following

Proposition 3.1. *For $0 < x_1 \le \cdots \le x_n$, $q = \min\{\omega_i\}, t \ge 0$, $f(t, \alpha)$ is a decreasing function of t for $\alpha \le (1 - q)^{-1}$ and $f(t, \alpha)$ is an increasing function for $\alpha \ge q^{-1}$.*

Proof. We will show the first assertion and the proof for the other one is similar. By Theorem 2.1 in [8], it suffices to prove the above result for $\alpha = (1 - q)^{-1}$. Let $f(t) = A^{(1-q)^{-1}}_{n,t} - G^{(1-q)^{-1}}_{n,t}$, it suffices to show $f'(0) \le 0$ which is equivalent to $A^q_n H^{1-q}_n \le G_n$, which is the weighted Sierpiński's inequality(See [7] for an extension of this) and this completes the proof. □

Theorem 3.1. *For $0 < q \le \min\{\omega_i\}$,*

$$x_1^{\frac{1}{1-q}-2}\sigma_n \ge 2(1-q)(A_n^{\frac{1}{1-q}} - G_n^{\frac{1}{1-q}}) \ge x_n^{\frac{1}{1-q}-2}\sigma_n \tag{3.1}$$

with equality holding if and only if $n = 2, q = 1/2$ or $x_1 = x_2 = \cdots = x_n$.

Proof. We prove the right-hand side inequality of (3.1) first. By homogeneity, we may assume $0 \le x_1 < x_2 < \cdots < x_n = 1$ in (3.1) and define

$$D_n(x_1, \cdots, x_{n-1}) = A_n^{\frac{1}{1-q}} - G_n^{\frac{1}{1-q}} - \sigma_n/2(1-q). \tag{3.2}$$

We want to show $D_n \ge 0$. Let $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in [0, 1]^{n-1}$ be the point in which the absolute minimum of D_n is reached.

We may assume $a_1 \le a_2 \le \cdots \le a_{n-1}$ and let $a_n = 1$. If $a_i = a_{i+1}$ for some $1 \le i \le n - 1$, by combining a_i with a_{i+1} and ω_i with ω_{i+1}, while noticing increasing q will decrease the value of $(1-q)(A_n^{\frac{1}{1-q}} - G_n^{\frac{1}{1-q}})$ by Lemma 2.1, we can reduce the determination of the absolute minimum of D_n to that of D_{n-1} with different weights. Thus without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$. If $a_1 > 0$ then $\mathbf{a}$ is an interior point of $[0, 1]^{n-1}$, then we obtain

$$\nabla D_n(a_1, \cdots, a_{n-1}) = 0$$

such that $a_1, \cdots, a_{n-1}$ solve the equation

$$x^2 - (A_n + A_n^{\frac{q}{1-q}})x + G_n^{\frac{1}{1-q}} = 0. \tag{3.3}$$

The above equation has at most two roots(regarding A_n, G_n as constants), so we are reduced to the case $n = 3$. But if $a_1 < a_2 < 1$ both satisfy (3.3), we will have

$$a_1a_2 = a_1^{\omega_1/(1-q)}a_2^{\omega_2/(1-q)},$$

which is impossible since $\omega_1+q\leq 1, \omega_2+q \leq 1$ and the two equalities can's hold at the same time. Thus if $a_1>0$, we only need to prove $D_2 \geq 0$. In this case, by letting $x=a_1>0$, we get

$$D_2(x)=(\omega_1x+\omega_2)^{\frac{1}{1-q}}-x^{\frac{\omega_1}{1-q}}-\frac{\omega_1\omega_2(x-1)^2}{2(1-q)}.$$

It's easy to check $D_2(1)=D_2'(1)=0$ and

$$\begin{aligned}\frac{1-q}{\omega_1}D''(x) &= \frac{q\omega_1}{1-q}(\omega_1x+\omega_2)^{\frac{2q-1}{1-q}}-(\frac{\omega_1}{1-q}-1)x^{\frac{\omega_1}{1-q}-2}-\omega_2 \\ &\geq \frac{q\omega_1}{1-q}+1-\frac{\omega_1}{1-q}-\omega_2=0.\end{aligned}$$

with equality holding if and only if $x=1$ or $q=1/2$. Hence by the Taylor expansion at 1, $D_2(x)\geq 0$ with equality holding if and only if $x=1$ or $q=1/2$.

If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1=0$, (3.2) is reduced to

$$E_n(x_1=0,\cdots,x_{n-1})=A_n^{\frac{1}{1-q}}-\sigma_n/2(1-q).$$

We now show $E_n\geq 0$. Let $(a_2,\cdots,a_{n-1})\in[0,1]^{n-2}$ be the point in which the absolute minimum of E_n is reached. Similar to the argument above, we may assume $0=a_1<a_2<\cdots<a_{n-1}<1$ and it's easy to show by using the method above that we only need to consider the cases $n=2$ and $n=3$. $E_2\geq 0$ is equivalent to $q^{1/(1-q)}\geq q/2$ and $g(q)=(1-q)^{1/(1-q)}-q/2\geq 0$. The first inequality follows from Lemma 2.2 and one checks $g(q)$ is a decreasing function of q hence $g(q)\geq g(1/2)=0$. For the case $n=3$, we set $x=a_2$ so that

$$\frac{1-q}{\omega_2}E_3'(a_2)=A_3^{\frac{q}{1-q}}-(a_2-A_3)=0.$$

Using this we get

$$\begin{aligned}\frac{(1-q)^2A_3}{\omega_2}E_3''(a_2) &= q\omega_2A_3^{\frac{q}{1-q}}-(1-q)(1-\omega_2)A_3 \\ &= q\omega_2(a_2-A_3)-(1-q)(1-\omega_2)A_3 \\ &= q\omega_2((1-\omega_2)a_2-\omega_3)-(1-q)(1-\omega_2)(\omega_2a_2+\omega_3) \\ &= \omega_2(1-\omega_2)(2q-1)a_2-q\omega_2\omega_3-(1-q)(1-\omega_2)\omega_3<0.\end{aligned}$$

This implies $E_3(x)$ takes its local maximum at a_2 so in order to show $E_3\geq 0$, we only need to show it for the cases $a_2=0$ or $a_2=1$ and we are then back to the case $n=2$ and this completes the proof for the right-hand side inequality of (3.1).

For the left-hand side inequality of (3.1), we may again assume $0\leq x_1<x_2<\cdots<x_n=1$ and define

$$F_n(x_1,\cdots,x_{n-1})=\sigma_n/2(1-q)-x_1^{2-\frac{1}{1-q}}(A_n^{\frac{1}{1-q}}-G_n^{\frac{1}{1-q}}). \tag{3.4}$$

We want to show $F_n\geq 0$. Let $\mathbf{a}=(a_1,\cdots,a_{n-1})\in[0,1]^{n-1}$ be the point in which the absolute minimum of F_n is reached.

Again we may assume $a_1<a_2<\cdots<a_{n-1}<1$. If $a_1>0$ then $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$, and we obtain

$$\nabla F_n(a_1,\cdots,a_{n-1})=0$$

such that $a_2,\cdots,a_{n-1}$ solve the equation $f(x)=0$ where

$$f(x)=x^2-(A_n+a_1^{2-\frac{1}{1-q}}A_n^{\frac{q}{1-q}})x+a_1^{2-\frac{1}{1-q}}G_n^{\frac{1}{1-q}}.$$

The above equation has at most two roots(regarding a_1,A_n,G_n as constants), so we are reduced to the case $n=4$. But note we also have $f(a_1)=\omega_1^{-1}(2-\frac{1}{1-q})a_1^{2-\frac{1}{1-q}}(A_n^{\frac{1}{1-q}}-G_n^{\frac{1}{1-q}})\geq 0$ and $f(1)\leq 0$ since otherwise by decreasing $a_n=1$, we will get a smaller value of F_n, contradicts to our assumption. Thus we only need to consider the case $n=3$. In this case a_2 is a root of $f(x)=0$ and the other root b satisfies $b \geq 1$ since $\lim_{x\to\infty} f(x)=\infty$. But then we will have

$$a_2 \leq ba_2 = a_1^{2-\frac{1}{1-q}} a_1^{\omega_1/(1-q)} a_2^{\omega_2/(1-q)},$$

which implies

$$a_1^{1-\omega_2/(1-q)} \leq a_2^{1-\omega_2/(1-q)} \leq a_1^{2-\frac{1}{1-q}} a_1^{\omega_1/(1-q)},$$

which is impossible. Thus if $a_1 > 0$, we only need to prove $F_2 \geq 0$ and this case can be proved similarly to our treatment of $D_2 \geq 0$.

If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, (3.4) follows trivially and this completes the proof for the left-hand side inequality of (3.1). □

Corollary 3.1. *For* $0 < q \leq \min\{\omega_i\}, 0 < x_1 \leq x_2 \leq \cdots \leq x_n < 1, x_1 \neq x_n$,

$$\left(\frac{1-x_1}{x_1}\right)^{2-\frac{1}{1-q}} \geq \frac{A_n^{\frac{1}{1-q}} - G_n^{\frac{1}{1-q}}}{A_n'^{\frac{1}{1-q}} - G_n'^{\frac{1}{1-q}}} \geq \left(\frac{1-x_n}{x_n}\right)^{2-\frac{1}{1-q}}.$$

Proof. Apply (3.1) to both $\mathbf{x}$, $\mathbf{x}'$ and take their quotients gives the desired result. □

Theorem 3.2. *For* $0 < q \leq \min\{\omega_i\}$,

$$x_n^{\frac{1}{q}-2}\sigma_n \geq 2q(A_n^{\frac{1}{q}} - G_n^{\frac{1}{q}}) \geq x_1^{\frac{1}{q}-2}\sigma_n \tag{3.5}$$

with equality holding if and only if $n=2, q=1/2$ *or* $x_1 = x_2 = \cdots = x_n$.

Proof. We prove the left-hand side inequality first. By homogeneity, we may assume $0 \leq x_1 < x_2 < \cdots < x_n = 1$ in (3.5) and define

$$D_n(x_1, \cdots, x_{n-1}) = \frac{1}{2q}\sigma_n - (A_n^{\frac{1}{q}} - G_n^{\frac{1}{q}}).$$

We want to show $D_n \geq 0$. Let $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of D_n is reached. As in the proof of Theorem 3.1 and again use Lemma 2.1, we may assume $a_1 < a_2 < \cdots < a_{n-1} < a_n = 1$. If $a_1 > 0$ then $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$, then we obtain

$$\nabla D_n(a_1, \cdots, a_{n-1}) = 0$$

such that $a_1, \cdots, a_{n-1}$ solve the equation

$$x^2 - (A_n + A_n^{\frac{1-q}{q}})x + G_n^{\frac{1}{q}} = 0. \tag{3.6}$$

The above equation has at most two roots(regarding A_n, G_n as constants), so we are reduced to the case $n=3$. But if $a_1 < a_2 < 1$ both satisfy (3.6), we will have

$$a_1 a_2 = a_1^{\omega_1/q} a_2^{\omega_2/q},$$

which is impossible since $\omega_1 \geq q, \omega_2 \geq q$ and the two equalities can's hold at the same time. Thus if $a_1 > 0$, we only need to prove $D_2 \geq 0$. In this case if $x = a_1 > 0$ and $\omega_1 = 1-q, \omega_2 = q$ then

$$g(x) := x + G_2^{\frac{1}{q}}/x = A_2^{\frac{1-q}{q}} + A_2.$$

Note for $x \leq u, q \leq 1/3$,

$$g'(u) = 1 - G_2^{\frac{1}{q}}/u^2 \geq g'(x) \geq 0,$$

since $0 < x < 1$ and $G_2 = x^{1-q}$. Since $x \leq A_2$ in our case, we then have $g(x) \leq g(A_2) = A_2 + G_2^{\frac{1}{q}}/A_2$, a contradiction.

Now suppose $q > 1/3$, then

$$D_2''(x) = \frac{1-q}{q^2}(q^2 - (1-q)^2 A_2^{\frac{1-2q}{q}} + (1-2q)x^{\frac{1-3q}{q}}) \geq \frac{1-q}{q^2}(q^2 - (1-q)^2 + (1-2q)) = 0,$$

with equality holding if and only if $q = 1/2$. As $D_2(1) = D_2'(1) = 0$, this shows $D_2(x) \geq 0$ by considering the Taylor expansion of D_2 at 1.

Now suppose $\omega_1 = q, \omega_2 = 1-q$, then $D_2''(x) = (1-q)(1 - A_2^{\frac{1-2q}{q}}) \geq 0$ with equality holding if and only if $q = 1/2$. As we also have $D_2(1) = D_2'(1) = 0$, this shows $D_2(x) \geq 0$.

Finally, we consider the case when D_n reaches its absolute minimum at $\mathbf{a}$ with $a_1 = 0$. Define

$$E_n(x_1 = 0, \cdots, x_{n-1}) = \frac{1}{2q}\sigma_n - A_n^{\frac{1}{q}}.$$

We show now $E_n \geq 0$. $E_2 \geq 0$ is equivalent to $g(q) = (1-q)/2 - q^{1/q} \geq 0$ and $(1-q)/2 - (1-q)^{1/q} \geq 0$, the second inequality follows from Lemma 2.2 and one checks $g(q)$ is a decreasing function of q so that $g(q) \geq g(1/2) = 0$ with equality holding if and only if $q = 1/2$.

Suppose now $n \geq 3$ and let $\mathbf{a} = (a_2, \cdots, a_{n-1}) \in [0,1]^{n-3}, 0 < a_2 < \cdots < a_{n-1} < 1$ be the point in which the absolute minimum of E_n is reached. Then

$$\nabla E_n(a_2, \cdots, a_{n-1}) = 0$$

such that $a_2, \cdots, a_{n-1}$ solve the equation

$$x - A_n - A_n^{\frac{1-q}{q}} = 0.$$

The above equation has at most one root(regarding A_n, G_n as constants). Thus it suffices to show $E_3 \geq 0$ under the condition $\omega_i \geq q$. Now $a_2 - A_3 = A_3^{\frac{1-q}{q}}$ and

$$\begin{aligned} E_3 &= \sum_{i=1}^{3} \omega_i (a_i - A_3)^2/2q - A_3^{1/q} \geq A_3^{\frac{2-2q}{q}} + A_3^2/2 - A_3^{1/q} \\ &\geq 2\sqrt{A_3^{\frac{2-2q}{q}} \cdot A_3^2/2} - A_3^{1/q} = (\sqrt{2}-1)A_3^{1/q} \geq 0. \end{aligned}$$

This completes the proof for the left-hand side inequality of (3.5) and for the right-hand side inequality of (3.5), we may again assume $0 \leq x_1 < x_2 < \cdots < x_n = 1$ and define

$$F_n(x_1, \cdots, x_n) = (A_n^{\frac{1}{q}} - G_n^{\frac{1}{q}}) - x_1^{\frac{1}{q}-2}\sigma_n/2q. \tag{3.7}$$

We want to show $F_n \geq 0$. Let $\mathbf{a} = (a_1, \cdots, a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of F_n is reached.

Again we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$. If $a_1 > 0$ then $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$, and we obtain

$$\nabla F_n(a_1, \cdots, a_{n-1}) = 0$$

such that $a_2, \cdots, a_{n-1}$ solve the equation $f(x) = 0$ where

$$f(x) = a_1^{\frac{1}{q}-2}x^2 - (A_n^{\frac{1}{q}-1} + a_1^{\frac{1}{q}-2}A_n)x + G_n^{\frac{1}{q}}.$$

The above equation has at most two roots(regarding a_1, A_n, G_n as constants), so we are reduced to the case $n = 4$. But note we also have $f(a_1) = -\omega_1^{-1}(\frac{1}{q} - 2)a_1^{\frac{1}{q}-2}\sigma_n \leq 0$ and $f(0) \geq 0$. Thus we only need to consider the case $n = 3$. In this case a_2 is a root of $f(x) = 0$ and the other root c satisfies $0 < c \leq a_1$. But then we will have

$$a_1 a_2 \geq c a_2 = a_1^{\frac{\omega_1}{q}} a_2^{\frac{\omega_2}{q}} a_1^{2-\frac{1}{q}},$$

which implies

$$a_2^{1-\omega_2/q} \geq a_1^{1+(\omega_1-1)/q},$$

which is impossible. Thus if $a_1 > 0$, we only need to prove $F_2 \geq 0$. By renormalizing $a_1 = 1, a_2 > 1$, this case follows similarly to our treatment of $D_2 \geq 0$.

If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, (3.7) follows trivially and this completes the proof for the right-hand side inequality of (3.5). □

The following corollary generalizes Theorem 1.2, the proof is similar to that of Corollary 3.1.

Corollary 3.2. *For $0 < q \leq \min\{\omega_i\}, 0 < x_1 \leq x_2 \leq \cdots \leq x_n < 1, x_1 \neq x_n$,*

$$(\frac{x_n}{1-x_n})^{\frac{1}{q}-2} \geq \frac{A_n^{\frac{1}{q}} - G_n^{\frac{1}{q}}}{A_n'^{\frac{1}{q}} - G_n'^{\frac{1}{q}}} \geq (\frac{x_1}{1-x_1})^{\frac{1}{q}-2}.$$

4. Some Inequalities among Symmetric Means

Let $s \in \{0, 1, \cdots, n\}$, the s-th symmetric function E_s of $\mathbf{x}$ and its mean P_s are defined by

$$E_s(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_s \leq n} \prod_{j=1}^{s} x_{i_j}, E_0 = 1; P_s(\mathbf{x}) = \frac{E_r(\mathbf{x})}{\binom{n}{2}}.$$

As mentioned in section 1, we shall write P_s for $P_s(\mathbf{x})$ and the meaning of $P'_s, P_{s,t}$ are similar. Theorem 1.2 can be generalized to inequalities involving the symmetric means.

Theorem 4.1. *For $n > 1, \omega_i = 1/n, 0 < x_1 \leq x_2 \leq \cdots \leq x_n, t \geq 0,\ 2 \leq r \leq n$.*

$$(4.1)\qquad (\frac{x_1}{1-x_1})^{r-2}(A_n'^r - P'_r) \leq \quad A_n^r - P_r \quad \leq (\frac{x_n}{1-x_n})^{r-2}(A_n'^r - P'_r),$$

$$(4.2)\qquad (\frac{x_1}{t+x_1})^{r-2}(A_{n,t}^r - P_{r,t}) \leq \quad A_n^r - P_r \quad \leq (\frac{x_n}{t+x_n})^{r-2}(A_{n,t}^r - P_{r,t}),$$

$$(4.3)\qquad \frac{r(r-1)x_1^{r-2}}{2(n-1)}\sigma_n \leq \quad A_n^r - P_r \quad \leq \frac{r(r-1)x_n^{r-2}}{2(n-1)}\sigma_n,$$

where we need $x_n < 1$ in (4.1).

Proof. We note (4.3) is a result of Dinghas[5], originally written as

$$\frac{r(r-1)x_1^{r-2}}{2n(n-1)}\sum_{k=1}^{n}(1-\frac{1}{k})(x_k - A_{k-1})^2 \leq A_n^r - P_r \leq \frac{r(r-1)x_n^{r-2}}{2n(n-1)}\sum_{k=1}^{n}(1-\frac{1}{k})(x_k - A_{k-1})^2.$$

By using the relation $(k-1)A_{k-1} + a_k = A_k$, one shows easily by induction that $\sum_{k=1}^{n}(1 - \frac{1}{k})(x_k - A_{k-1})^2 = n\sigma_n$ and (4.3) then follows. Applying (4.3) to both $A_n^r - P_r$ and $A_n'^r - P'_r$ and take their quotients, we obtain (4.1). To show (4.2), we use another identity of Dinghas[5]:

$$(4.4)\qquad A_n^r - P_r = \frac{\binom{n-2}{r-s}}{\binom{n}{r}}\sum_{k=2}^{n}\sum_{i=2}^{k}(i-1)\frac{(x_k - A_{k-1})^2}{k^2}P_{r-2,n-2}^{i-2,k-i}(A_{k-1}; A_k; x_{k+1}, \cdots, x_n)$$

where $P_{r-2,n-2}^{i-2,k-i}(A_{k-1}; A_k; x_{k+1}, \cdots, x_n)$ denotes the $(r-2)$-th symmetric mean of the $n-2$ numbers $A_{k-1}(i-2 \text{ times})$, $A_k(k-i \text{ times})$ and $x_{k+1}, \cdots, x_n$.

Now use (4.4) for $(A_n^r - P_r)/x_n^{r-2}$ and $(A_{n,t}^r - P_{r,t})/(x_n+t)^{r-2}$ and consider their differences, the right-hand side inequality of (4.2) follows from this and the observation

$$\frac{x_i}{x_n} \leq \frac{x_i + t}{x_n + t}, 1 \leq i \leq n; \frac{A_i}{x_n} \leq \frac{A_{i,t}}{x_n + t}, i = k-1, k.$$

The left-hand side inequality of (4.2) can be shown similarly and this completes the proof. □

We note here (4.2) also implies (4.3). This can be seen by noticing $\lim_{t\to\infty}((x_n+t)^{2-r}(A^r_{n,t}-P_{r,t})=r(r-1)\sigma_n/2(n-1)$.

Corollary 4.1. *For $r\geq 2$,*

$$rx_1(A_n^{r-1}-P_{r-1})\leq (r-2)(A_n^r-P_r)\leq rx_n(A_n^{r-1}-P_{r-1}). \tag{4.5}$$

Proof. Let $f(t)=x_{n,t}^{2-r}(A^r_{n,t}-P_{r,t})$. By (4.2), f is an increasing function of t and $f'(0)\geq 0$ gives the right-hand inequality of (4.5) and the left-hand inequality of (4.5) follows similarly. □

Theorem 4.2. *For $n>1,\omega_i=1/n, 0<x_1\leq x_2\leq\cdots\leq x_n, t\geq 0$, $1\leq r\leq n-1$.*

$$\begin{aligned}(\frac{x_1}{1-x_1})^{2r-2}(P_r'^2-P'_{r-1}P'_{r+1})\leq\ & P_r^2-P_{r-1}P_{r+1} & \leq (\frac{x_n}{1-x_n})^{2r-2}(P_r'^2-P'_{r-1}P'_{r+1}),\\ (\frac{x_1}{t+x_1})^{2r-2}(P_{r,t}^2-P_{r-1,t}P_{r+1,t})\leq\ & P_r^2-P_{r-1}P_{r+1} & \leq (\frac{x_n}{t+x_n})^{2r-2}(P_{r,t}^2-P_{r-1,t}P_{r+1,t}),\\ \frac{x_1^{2r-2}}{(n-1)}\sigma_n\leq\ & P_r^2-P_{r-1}P_{r+1} & \leq \frac{x_n^{2r-2}}{(n-1)}\sigma_n,\end{aligned} \tag{4.6}$$

where we need $x_n<1$ in (4.6).

Proof. The proof is similar to the proof of Theorem 4.1, once we note the following identity of Muirhead[14](see also [11], Theorem 54).

$$P_r^2-P_{r-1}P_{r+1}=(r(r+1)\binom{n}{r}\binom{n}{r+1})^{-1}\sum_{i=0}^{r-1}\binom{2i}{i}\frac{(r,i)}{i+1},$$

where $(r,i)=\sum x_1^2\cdots x_{r-i-1}^2x_{r-i}\cdots x_{r+i-1}(x_{r+i}-x_{r+i+1})^2$, the summation extending over all products formed from the $\mathbf{x}$ and of the type shown. □

We leave the proof of the following corollary to the reader since it is similar to the one of Corollary 4.1.

Corollary 4.2. *For $2\leq r\leq n-1$,*

$$x_1(P_rP_{r-1}-P_{r-2}P_{r+1})\leq 2(P_r^2-P_{r-1}P_{r+1})\leq x_n(P_rP_{r-1}-P_{r-2}P_{r+1}).$$

5. Further Discussions

Theorem 5.1. *For $-1\leq r\neq 1\leq 2$,*

$$|A_n-M_{n,r}|\geq\frac{|1-r|\sigma_n}{2(dx_n+(1-d)x_1)}, \tag{5.1}$$

For $-1/2\leq r<1$,

$$A_n-M_{n,r}\leq(\frac{d}{x_1}+\frac{1-d}{x_n})\frac{(1-r)\sigma_n}{2}, \tag{5.2}$$

where $d=\max\{(2-r)/3,(1+r)/3\}$ and equality hold in both cases if and only if $x_1=\cdots=x_n$.

Proof. A close look at the proof of Theorem 3.1 in [8] shows that the first inequality holds. Similarly to the argument in the proof of Theorem 3.1 in [8], the proof of (5.2) can be reduced to the case $n=2$. By setting $0<x_1=x\leq x_2=1,\omega_1=q,\omega_2=1-q$, $f(x)=x(qx+1-q-(qx^r+1-q)^{1/r})-(1-r)q(1-q)(d+(1-d)x)(x-1)^2/2$. We need to show $f(x)\leq 0$ for $-1/2\leq r<1$. It's easy to check that $f(1)=f'(1)=f''(1)=0$ and

$$f'''(x)=q(1-q)(1-r)[(q+(1-q)x^{-r})^{\frac{1-3r}{r}}x^{-r-1}((1-q)(1+r)x^{-r}+q(2-r))-3(1-d)].$$

Note $(q+(1-q)x^{-r})^{\frac{1-3r}{r}}x^{-r-1}=(qx^r+(1-q))^{\frac{1-3r}{r}}x^{2r-2}\geq 1$ for $-1\leq r<1$. For $0\leq r\leq 1/2$, $(1-q)(1+r)x^{-r}+q(2-r)\geq 1+r+(1-2r)q\geq r+1$ and for $1/2<r<1$, $(1-q)(1+r)x^{-r}+q(2-r)\geq 1+r+(1-2r)q\geq 2-r$, (5.2) holds for our choice of d. When $-1/2\leq r<0$, we write $f'''(x)$ as

$$f'''(x)=q(1-q)(1-r)[(q+(1-q)x^{-r})^{\frac{1-3r}{r}}x^{-2r-1}((1-q)(1+r)+q(2-r)x^r)-3(1-d)],$$

and the conclusion follows similarly. □

We note here when $r=0$, (5.2) implies (5.1). By writing $f(t)=(d(x_n+t)+(1-d)(x_1+t))(A_{n,t}-G_{n,t})$ and noticing $\lim_{t\to\infty}f(t)=\sigma_n/2$, it suffices to show $f'(t)\leq 0$ or equivalently $A_n-G_n+(dx_n+(1-d)x_1)(1-G_n/H_n)\leq 0$ since $\mathbf{x}$ is arbitrary. Now by repeating the same method we see that (5.2) implies (5.1).

We end this paper by proving the following theorem, part of which was a conjecture of the author in [8].

Theorem 5.2. *For* $0<x_1\leq\cdots\leq x_n$, $q=\min\{\omega_i\}$

$$((1-q)/2x_1+q/2x_n)\sigma_n\geq(A_n-G_n)\geq\sigma_n/2((1-q)x_n+qx_1)$$

Proof. For the right-hand side inequality, (5.1) shows

$$2(2x_n+x_1)(A_n-G_n)\geq 3\sigma_n.$$

Thus when $q\leq 1/3$ we are done. But if $q>1/3$, one must have $n=2$ and one checks by direct calculation(see the proof of Theorem 3.1 in [8], replacing c by $2q$ there) that the above conjecture holds for $n=2$. The proof for the left-hand side inequality is similar. □

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