ADMISSIBILITY AND NON-UNIFORM DICHOTOMY FOR DIFFERENTIAL SYSTEMS

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Abstract. The problem of nonuniform exponential dichotomy of linear differential systems in Banach spaces is discussed. It is established a connection between the admissibility of a pair of certain function spaces which are translations invariant, on one hand, and the nonuniform exponential dichotomy of differential systems, on the other. Also, Some results due to Hartman, Massera, Schäffer and Coppel are generalized as well.

1. Introduction

In the last few decades, much significant development on the classical ideas of J. Daleckij and M. Krein [4] and J. L. Massera and J. J. Schäffer [10, 11] on exponential dichotomy and other asymptotic properties concerning the solutions of differential equations has been witnessed. This is due to the many different applications in the theory of partial differential equations, probability theory, mathematical physics, and other areas, and also to the development of new techniques. One important technique is given in the paper "Die stabilitätsfrage bei differentialgeighungen" [17], where Perron gave a characterization of the exponential stability of the solutions to the linear differential equations

$$\frac{dx}{dt} = A(t)x, \quad t \in [0, +\infty), \quad x \in \mathbb{R}^n,$$

where $A(t)$ is a matrix bounded continuous function, in terms of the existence of bounded solutions of the equations $\frac{dx}{dt} = A(t)x + f(t)$, where f is a continuous bounded function on $\mathbb{R}_+$. It played an important role in the early development of the qualitative theory of differential systems. After the seminal researches of O. Perron, relevant results concerning the extension of Perron's problem in the more general framework of infinite-dimensional Banach spaces were obtained by M. G. Krein, J. L. Daleckij, R. Bellman, J. L. Massera and J. J. Schäffer, P. Hartman, W.A. Coppel.

Firstly, J. L. Massera and J. J. Schäffer have obtained in [10] results for the behavior of solutions of the homogenous differential equations imitating a non-uniform exponential dichotomy, and later, this case has been studied by M. Reghiş [19], V. A. Pliss [15], P. Preda [16, 17] and others.

The aim of this paper is to give sufficient conditions for exponential dichotomy of differential systems in the more general case when the coefficients of the differential systems are not necessarily integral bounded, as can be seen in the first section below. Also some well-known results given by J.L. Massera and J.J. Schäffer in [10] are extended.

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2. Preliminaries

We begin by recalling some standard notations and definitions.

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded, linear operators acting on X and $\mathbb{R}_+ = [0, \infty)$ with $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$.

Also, we denote by $\mathcal{M}(\mathbb{R}_+, X)$ the space of all Bochner measurable functions from $\mathbb{R}_+$ to X and by:

$$L^1_{loc}(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X) : \int\limits_K ||f(t))||dt < \infty, \text{ for each compact } K \text{ in } \mathbb{R}_+ \right\};$$

$$L^p(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X) : \int\limits_{\mathbb{R}_+} ||f(t)||^p dt < \infty \right\}, \text{ where } p \in [1, \infty);$$

$$L^\infty(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X) : ess \sup_{t \in \mathbb{R}_+} ||f(t)|| < \infty \right\};$$

and

$$M^p(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X) : \sup_{t \in \mathbb{R}_+} \int\limits_t^{t+1} ||f(s)||^p ds < \infty \right\} \text{ where, } p \in [1, \infty).$$

We note that $L^p(\mathbb{R}_+, X)$, $L^\infty(\mathbb{R}_+, X)$, $M^p(\mathbb{R}_+, X)$ are Banach spaces endowed with the respectively norms:

$$||f||_p = \left(\int\limits_{\mathbb{R}_+} ||f(t)||^p dt \right)^{\frac{1}{p}},$$

$$||f||_\infty = ess \sup_{t \in \mathbb{R}_+} ||f(t)||,$$

and

$$||f||_{M^p} = \sup_{t \in \mathbb{R}_+} \left(\int\limits_t^{t+1} ||f(s)||^p ds \right)^{\frac{1}{p}}.$$

In order to simplify the notations we put $L^p := L^p(\mathbb{R}_+, X)$, $L^\infty := L^\infty(\mathbb{R}_+, X)$, $M^p := M^p(\mathbb{R}_+, X)$, for all $p \in [1, \infty)$ and $L^1_{loc} = L^1_{loc}(\mathbb{R}_+, X)$.

For the operator valued function $A : \mathbb{R}_+ \to \mathcal{M}(X)$, that is locally Bochner integrable on $\mathbb{R}_+$ we consider the homogeneous differential system:

$$(A) \qquad x'(t) = A(t)x(t),$$

and the associated inhomogeneous differential system:

$$(A, f) \qquad x'(t) = A(t)x(t) + f(t),$$

where $f : \mathbb{R}_+ \to X$ is locally integrable on $\mathbb{R}_+$. It is well-known that the system (A) is said to have integral bounded coefficients if the operator valued function $t \longmapsto A(t)$ is in $M^1(\mathbb{R}_+, \mathcal{B}(X))$ (i.e. $\sup\limits_{t \geq 0} \int_t^{t+1} \|A(u)\| du < \infty$).

In what follows we will denote by U the unique solution of the Cauchy Problem:

$$\begin{cases} U'(t) = A(t)U(t); \\ U(0) = I, \end{cases}$$

where I denotes the identity on X. It is known (see for instance [4]) that for each $t \in \mathbb{R}_+$, the operator $U(t)$ is an invertible operator with:

$$\begin{cases} U'^{-1}(t) = -U^{-1}(t)A(t); \\ U^{-1}(0) = I. \end{cases}$$

We also suppose that $X_0 = \{x \in X : U(\cdot)x \in L^\infty\}$ is a closed linear subspace of X and that there exists X_1 a complement of X_0. In the hypothesis we denote by P_1, P_2 the projectors on X_0, respectively on X_1.

Definition 1. *The differential system (A) is said to be non-uniformly exponentially dichotomic if there exists a positive function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a positive constant ν such that the following conditions hold:*

$$(E_1^n) \qquad \|U(t)P_1x\| \leq N(t_0)e^{-\nu(t-t_0)}\|U(t_0)P_1x\|$$

$$(E_2^n) \qquad \|U(t)P_2x\| \geq \frac{1}{N(t)}e^{\nu(t-t_0)}\|U(t_0)P_2x\|,$$

for all $t \geq t_0 \geq 0$ and all $x \in X$

We note that if $X_0 = X$ then the condition (E_1^n) from the definition above is equivalent with:

$$(E^n) \qquad \|U(t)U^{-1}(t_0)\| \leq N(t_0)e^{-\nu(t-t_0)}$$

for all $t \geq t_0 \geq 0$, and the system (A), which satisfies (E^n), will be called *non-uniformly exponentially stable.*

Remark 1. *The system (A) is non-uniformly exponentially stable if and only if there exist two strictly positive constants N, ν such that the following statement holds:*

$$\|U(t)\| \leq Ne^{-\nu t}, \qquad \textit{for all}\, t \geq 0.$$

For $p \in [1, \infty]$ we will denote by

$$q = \begin{cases} \frac{p}{p-1}, & p \in (1, \infty) \\ \infty, & p = 1 \\ 1, & p = \infty. \end{cases}$$

Definition 2. *The pair (L^p, L^∞) is said to be admissible to (A) if for each $f \in L^p$, there exists a solution $x \in L^\infty$ of the inhomogeneous system (A, f).*

For more convenience we will recall the following known result (see for instance [11]):

Theorem 1. *If the pair (L^p, L^∞) is admissible to (A) then there exists a positive constant K such that for each $f \in L^p$, there exists a unique solution $x \in L^\infty$ for the equation (A, f), with $x(0) \in X_1$ and $\|x\|_\infty \leq K\|f\|_p$.*

In what follows, we will denote by $\varphi_{[a,b]}$ the characteristic function (indicator) of the interval $[a, b]$.

3. Non-uniform Dichotomy

Now we can state the main result of our paper:

Theorem 2. *If there exists $p \in (1, \infty]$ such that the pair (L^p, L^∞) is admissible to (A), then there exists a function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\nu > 0$ such that the following statements hold:*

(i) $\|U(t)P_1x\| \le N(t_0)e^{-\nu(t-t_0)^{\frac{1}{q}}}\|U(t_0)P_1x\|$

(ii) $\|U(t)P_2x\| \ge N^{-1}(t)e^{\nu(t-t_0)^{\frac{1}{q}}}\|U(t_0)P_2x\|$,
for all $t \ge t_0 \ge 0$, and all $x \in X$.

Proof. Let $0 \le t_0 < t_1$ and $x \in X$ with $P_1x \ne 0$.

Also, we set

$$x(t) = U(t)P_1x \int_0^t \frac{\varphi_{[t_0,t_1]}(\tau)}{\|U(\tau)P_1x\|}d\tau.$$

Then $x \in L^\infty$ and:

$$x'(t) = A(t)x(t) + \varphi_{[t_0,t_1]}(t)\frac{U(t)P_1x}{\|U(t)P_1x\|},$$
$$x(0) = 0 \in X_1.$$

So there exists a positive constant K such that:

$$\|U(t)P_1x\| \int_0^t \frac{\varphi_{[t_0,t_1]}(\tau)}{\|U(\tau)P_1x\|}d\tau \le K(t_1 - t_0)^{\frac{1}{p}}, \text{ for all } t \ge 0 \tag{3.1}$$

Putting $t_1 = t$ in the above relation (3.1) we can deduce that

$$\|U(t)P_1x\| \int_{t_0}^t \frac{d\tau}{\|U(\tau)P_1x\|} \le K(t - t_0)^{\frac{1}{p}} \tag{3.2}$$

for all $t \ge t_0 \ge 0$.

Denoting by $\varphi(t) = \int_{t_0}^t \frac{d\tau}{\|U(\tau)P_1x\|}$ and using the relation (3.2) we obtain that:

$$\varphi(t) \le K(t - t_0)^{\frac{1}{p}}\varphi'(t) \tag{3.3}$$

for all $t \ge t_0 \ge 0$, which implies that

$$\frac{1}{K}(t - t_0)^{\frac{-1}{p}} \le \frac{\varphi'(t)}{\varphi(t)} \tag{3.4}$$

for all $t \ge t_0 \ge 0$, so we have that:

$$\varphi(t_0 + 1)e^{\frac{-q}{k}}e^{\frac{q}{k}(t-t_0)^{\frac{1}{q}}} \le \varphi(t) \le K(t - t_0)^{\frac{1}{p}}\frac{1}{\|U(t)P_1x\|}, \tag{3.5}$$

for all $t \ge t_0 + 1$.

However,

$$\|U(\tau)P_1x\| \le \|U(t_0)P_1x\|e^{\int_{t_0}^{\tau}\|A(u)\|du} \le \|U(t_0)P_1x\|e^{\int_{t_0}^{t_0+1}\|A(u)\|du},$$

for all $\tau \in [t_0, t_0 + 1]$, which implies that

$$\varphi(t_0 + 1) \ge \|U(t_0)P_1x\|^{-1}e^{-\int_{t_0}^{t_0+1}\|A(u)\|du} \tag{3.6}$$

and thus

$$\|U(t)P_1x\| \le e^{\frac{q}{K}}K(t - t_0)^{\frac{1}{p}}e^{\frac{-q}{k}(t-t_0)^{\frac{1}{q}}}e^{\int_{t_0}^{t_0+1}\|A(u)\|du}\|U(t_0)P_1x\|, \tag{3.7}$$

for all $t \ge t_0 + 1$.

Denoting by $u := t - t_0 \geq 1$ and $\nu = \frac{q}{2K}$, we observe that

$$\sup_{u \geq 1} u^{\frac{1}{p}} e^{-\nu u^{\frac{1}{q}}} = M > 0$$

and from (3.7) we obtain

$$\|U(t)P_1x\| \leq e^{\frac{q}{K}} KMe^{\int_{t_0}^{t_0+1} \|A(u)\|du} e^{-\nu(t-t_0)^{\frac{1}{q}}} \|U(t_0)P_1x\|, \tag{3.8}$$

for all $t \geq t_0 + 1$.

If $t \in [t_0, t_0 + 1]$, then we have:

$$\begin{aligned}\|U(t)P_1x\| &\leq \|U(t_0)P_1x\| e^{-\nu(t-t_0)^{\frac{1}{q}}} e^{\nu(t-t_0)^{\frac{1}{q}}} e^{\int_{t_0}^{t_0+1} \|A(u)\|du} \\ &\leq e^{\int_{t_0}^{t_0+1} \|A(u)\|du} e^{\nu} e^{-\nu(t-t_0)^{\frac{1}{q}}}.\end{aligned}$$

So we have that

$$\|U(t)P_1x\| \leq \max\{KMe^{2\nu}, e^{\nu}\} e^{\int_{t_0}^{t_0+1} \|A(u)\|du} e^{-\nu(t-t_0)^{\frac{1}{q}}}$$

for all $t \geq t_0 \geq 0$.

If we denote by:

$$N(t_0) = \max\{KMe^{2\nu}, e^{\nu}\} \cdot e^{\int_{t_0}^{t_0+1} \|A(u)\|du} \tag{3.9}$$

we deduce that

$$\|U(t)P_1x\| \leq N(t_0) e^{-\nu(t-t_0)^{\frac{1}{q}}} \|U(t_0)P_1x\| \tag{3.10}$$

for all $t \geq t_0 \geq 0$.

In the case when $P_2x \neq 0$ we will denote by

$$y(t) = U(t)P_2x \int_t^\infty \frac{\varphi_{[t_0,t_1]}(\tau)}{\|U(\tau)P_2x\|} d\tau,$$

which is a solution of the differential system

$$y'(t) = A(t)y(t) - \varphi_{[t_0,t_1]}(t) \frac{U(t)P_2x}{\|U(t)P_2x\|} \text{ with } y(0) \in X_1 \text{ and } y \in L^\infty.$$

Then

$$\|U(t)P_2x\| \int_t^\infty \frac{\varphi_{[t_0,t_1]}(\tau)}{\|U(\tau)P_2x\|} d\tau \leq K(t_1 - t_0)^{\frac{1}{p}} \tag{3.11}$$

for all $t \geq 0$.

If $t = t_0$ in (3.11), then we can observe that

$$\|U(t)P_2x\| \int_t^{t_1} \frac{d\tau}{\|U(\tau)P_2x\|} \leq K(t_1 - t)^{\frac{1}{p}} \tag{3.12}$$

for all $t \leq t_1$. This implies that

$$\|U(s)P_2x\| \int_s^{t+1} \frac{d\tau)}{\|U(\tau)P_2x\|} \leq K(t + 1 - s) \tag{3.13}$$

for all $s \leq t + 1$.

Denoting by

$$\psi(s) = \int_s^{t+1} \frac{d\tau)}{\|U(\tau)P_2x\|}$$

it follows that

$$\psi(s) \leq -K(t+1-s)^{\frac{1}{p}}\psi'(s)$$

and hence

$$-\frac{1}{K}(t+1-\tau)^{-\frac{1}{p}} \geq \frac{\psi'(\tau)}{\psi(\tau)}, \qquad \text{for all } \tau \in [0, t+1)$$

and so

$$\int_s^t -\frac{1}{K}(t+1-\tau)^{-\frac{1}{p}} d\tau \geq \ln \frac{\psi(t)}{\psi(s)}, \qquad \text{for all } s \leq t.$$

From this it follows that

$$\psi(t) \leq \psi(s) e^{\frac{q}{K}} e^{-\frac{q}{K}(t+1-s)^{\frac{1}{q}}} \leq K(t+1-s)^{\frac{1}{p}} \frac{1}{\|U(s)P_2x\|} e^{\frac{q}{K}} e^{-\frac{q}{K}(t+1-s)^{\frac{1}{q}}} \tag{3.14}$$

for all $s \leq t$.

However, for $\tau \in [t, t+1]$ we have that:

$$\|U(\tau)P_2x\| \leq \|U(t)P_2x\| e^{\int_t^{t+1} \|A(u)\| du},$$

which implies that

$$\psi(t) \geq \frac{1}{\|U(t)P_2x\|} e^{-\int_t^{t+1} \|A(u)\| du}$$

and using (3.14) we obtain

$$\frac{1}{\|U(t)P_2x\|} e^{-\int_t^{t+1} \|A(u)\| du} \leq K(t+1-s)^{\frac{1}{p}} \frac{1}{\|U(s)P_2x\|} e^{\frac{q}{K}} e^{-\frac{q}{K}(t+1-s)^{\frac{1}{q}}} \tag{3.15}$$

and hence

$$\|U(s)P_2x\| e^{-\frac{q}{K}} e^{\frac{q}{K}(t+1-s)^{\frac{1}{q}}} \frac{1}{K}(t+1-s)^{-\frac{1}{p}} e^{-\int_t^{t+1} \|A(u)\| du} \leq \|U(t)P_2x\| \tag{3.16}$$

for all $s \leq t$.

If we denote by

$$\alpha = \inf_{u \geq 1} e^{\nu u^{\frac{1}{q}}} u^{-\frac{1}{p}} = \inf_{u \geq 1} (u^{\frac{1}{p}} e^{-\nu u^{\frac{1}{q}}})^{-1} = \frac{1}{\sup\limits_{u \geq 1} u^{\frac{1}{p}} e^{-\nu u^{\frac{1}{q}}}} = \frac{1}{M},$$

from (3.16) we obtain that

$$\frac{1}{e^{2\nu} K M e^{\int_t^{t+1} \|A(u)\| du}} e^{\nu(t+1-s)^{\frac{1}{q}}} \|U(s)P_2x\| \leq \|U(t)P_2x\|,$$

for all $t \geq s \geq 0$. This implies that:

$$\frac{e^{\nu(t-s)^{\frac{1}{q}}} \|U(s)P_2x\|}{\max\{e^{2\nu} K M, e^{\nu}\} \cdot e^{\int_t^{t+1} \|A(u)\| du}} \leq \|U(t)P_2x\|, \tag{3.17}$$

which is equivalent to (3.9) and $s := t_0$ with:

$$\|U(t)P_2x\| \geq \frac{1}{N(t)} e^{\nu(t-t_0)} \|U(t_0)P_2x\|,$$

for all $t \geq t_0 \geq 0$ and all $x \in X$. Thus the proof is completed. ∎

As a consequence we can remark that:

Corollary 1. *If the pair (L^∞, L^∞) is admissible to (A), then the system (A) is non-uniformly exponentially dichotomic.*

We note that if $X_0 = X$ then by Theorem 2, the result given in [10, Theorem 5.1, page 534] can be obtained as a particular case. Also, in the case when the coefficients of (A) are integrally bounded (i.e. $\sup\limits_{t\geq 0}\int_t^{t+1}\|A(u)\|du < \infty$), then the function N from Theorem 2 is bounded and this can be found as a particular case of the result from [10, Theorem 5.3, page 539].

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