REVERSES OF THE SCHWARZ INEQUALITY IN INNER PRODUCT SPACES AND APPLICATIONS

SEVER S. DRAGOMIR

ABSTRACT. Recent reverses for the celebrated Schwarz inequality in inner product spaces and applications for discrete and integral inequalities are surveyed.

1. INTRODUCTION

Let H be a linear space over the real or complex number field $\mathbb{K}$. The functional $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ is called an *inner product* on H if it satisfies the conditions

(i) $\langle x,x\rangle \geq 0$ for any $x \in H$ and $\langle x,x\rangle = 0$ iff $x = 0$;
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x,z\rangle + \beta \langle y,z\rangle$ for any $\alpha, \beta \in \mathbb{K}$ and $x,y,z \in H$;
(iii) $\langle y,x\rangle = \overline{\langle x,y\rangle}$ for any $x,y \in H$.

A first fundamental consequence of the properties (i)-(iii) above, is the *Schwarz inequality:*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle \langle y,y\rangle, \tag{1.1}$$

for any $x,y \in H$. The equality holds in (1.1) if and only if the vectors x and y are *linearly dependent,* i.e., there exists a nonzero constant $\alpha \in \mathbb{K}$ so that $x = \alpha y$.

If we denote $\|x\| := \sqrt{\langle x,x\rangle}, x \in H$, then one may state the following properties

(n) $\|x\| \geq 0$ for any $x \in H$ and $\|x\| = 0$ iff $x = 0$;
(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any $\alpha \in \mathbb{K}$ and $x \in H$;
(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for any $x,y \in H$ (the triangle inequality);

i.e., $\|\cdot\|$ is a *norm* on H.

In this survey paper we present some recent reverse inequalities for the Schwarz and the triangle inequalities. More precisely, we point out upper bounds for the nonnegative quantities

$$\|x\| \|y\| - |\langle x,y\rangle|, \quad \|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2$$

and

$$\|x\| + \|y\| - \|x + y\|$$

under various assumptions for the vectors $x,y \in H$.

If the vectors $x,y \in H$ are not *orthogonal*, i.e., $\langle x,y\rangle \neq 0$, then some upper bounds for the supra-unitary quantities

$$\frac{\|x\| \|y\|}{|\langle x,y\rangle|}, \quad \frac{\|x\|^2 \|y\|^2}{|\langle x,y\rangle|^2}$$

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are provided as well.

Applications for discrete and integral inequalities are also pointed out.

2. An Additive Reverse of the Schwarz Inequality

2.1. Introduction.

Let $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ be two positive $n-$*tuples* with

$$0 < m_1 \le a_i \le M_1 < \infty \text{ and } 0 < m_2 \le b_i \le M_2 < \infty; \tag{2.1}$$

for each $i \in \{1, \dots, n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

(1) *Pólya-Szegö's inequality* [20]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2.$$

(2) *Shisha-Mond's inequality* [23]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \le \left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

(3) *Ozeki's inequality* [19]

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k\right)^2 \le \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2.$$

(4) *Diaz-Metcalf's inequality* [1]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1}\sum_{k=1}^n a_k^2 \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^n a_k b_k.$$

If $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

(1) *Cassel's inequality* [24]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition

$$0 < m \le \frac{a_k}{b_k} \le M < \infty \text{ for each } k \in \{1, \dots, n\}, \tag{2.2}$$

then

$$\frac{\left(\sum_{k=1}^n w_k a_k^2\right)\left(\sum_{k=1}^n w_k b_k^2\right)}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \le \frac{(M+m)^2}{4mM}.$$

(2) *Greub-Reinboldt's inequality* [12]. We have

$$\left(\sum_{k=1}^n w_k a_k^2\right)\left(\sum_{k=1}^n w_k b_k^2\right) \le \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}\left(\sum_{k=1}^n w_k a_k b_k\right)^2,$$

provided $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition (2.1).

(3) *Generalised Diaz-Metcalf's inequality* [1], see also [17, p. 123]. If $u, v \in [0,1]$ and $v \le u$, $u+v=1$ and (2.2) holds, then one has the inequality

$$u \sum_{k=1}^{n} w_k b_k^2 + vMm \sum_{k=1}^{n} w_k a_k^2 \le (vm + uM) \sum_{k=1}^{n} w_k a_k b_k.$$

(4) *Klamkin-McLenaghan's inequality* [15]. If $\overline{\mathbf{a}}, \overline{\mathbf{b}}$ satisfy (2.2), then

$$\left(\sum_{i=1}^{n} w_i a_i^2\right)\left(\sum_{i=1}^{n} w_i b_i^2\right) - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \le \left(M^{\frac{1}{2}} - m^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2. \tag{2.3}$$

For other recent results providing discrete reverse inequalities, see the recent survey online [4].

In this section, by following [5], we point out a new reverse of Schwarz's inequality in real or complex inner product spaces. Particular cases for isotonic linear functionals, integrals and sequences are also given.

2.2. **An Additive Reverse Inequality.** The following reverse of Schwarz's inequality in inner product spaces holds [5].

Theorem 1. *Let $A, a \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $x, y \in H$. If*

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \ge 0, \tag{2.4}$$

or equivalently,

$$\left\| x - \frac{a+A}{2} \cdot y \right\| \le \frac{1}{2} |A - a| \|y\|, \tag{2.5}$$

holds, then one has the inequality

$$0 \le \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \le \frac{1}{4} |A - a|^2 \|y\|^4. \tag{2.6}$$

The constant $\frac{1}{4}$ is sharp in (2.6).

Proof. The equivalence between (2.4) and (2.5) can be easily proved, see for example [3].

Let us define

$$I_1 := \operatorname{Re}\left[\left(A \|y\|^2 - \langle x, y \rangle\right)\left(\overline{\langle x, y \rangle} - \overline{a} \|y\|^2\right)\right]$$

and

$$I_2 := \|y\|^2 \operatorname{Re} \langle Ay - x, x - ay \rangle.$$

Then

$$I_1 = \|y\|^2 \operatorname{Re}\left[A\overline{\langle x, y \rangle} + \overline{a} \langle x, y \rangle\right] - |\langle x, y \rangle|^2 - \|y\|^4 \operatorname{Re}(A\overline{a})$$

and

$$I_2 = \|y\|^2 \operatorname{Re}\left[A\overline{\langle x, y \rangle} + \overline{a} \langle x, y \rangle\right] - \|x\|^2 \|y\|^2 - \|y\|^4 \operatorname{Re}(A\overline{a}),$$

which gives

$$I_1 - I_2 = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2,$$

for any $x, y \in H$ and $a, A \in \mathbb{K}$. This is an interesting identity in itself as well.

If (2.4) holds, then $I_2 \geq 0$ and thus

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \operatorname{Re}\left[\left(A\|y\|^2 - \langle x, y\rangle\right)\left(\overline{\langle x, y\rangle} - \overline{a}\|y\|^2\right)\right]. \tag{2.7}$$

Further, if we use the elementary inequality for $u, v \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$)

$$\operatorname{Re}(u\overline{v}) \leq \frac{1}{4}|u+v|^2,$$

then we have, for

$$u := A\|y\|^2 - \langle x, y\rangle, \quad v := \langle x, y\rangle - a\|y\|^2,$$

that

$$\operatorname{Re}\left[\left(A\|y\|^2 - \langle x, y\rangle\right)\left(\overline{\langle x, y\rangle} - \overline{a}\|y\|^2\right)\right] \leq \frac{1}{4}|A-a|^2\|y\|^4. \tag{2.8}$$

Making use of the inequalities (2.7) and (2.8), we deduce (2.6).

Now, assume that (2.6) holds with a constant $C > 0$, i.e.,

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq C|A-a|^2\|y\|^4, \tag{2.9}$$

where x, y, a, A satisfy (2.4).

Consider $y \in H$, $\|y\| = 1$, $a \neq A$ and $m \in H$, $\|m\| = 1$ with $m \perp y$. Define

$$x := \frac{A+a}{2}y + \frac{A-a}{2}m.$$

Then

$$\langle Ay - x, x - ay\rangle = \left|\frac{A-a}{2}\right|^2 \langle y - m, y + m\rangle = 0,$$

and thus the condition (2.4) is fulfilled. From (2.9) we deduce

$$\left\|\frac{A+a}{2}y + \frac{A-a}{2}m\right\|^2 - \left|\left\langle\frac{A+a}{2}y + \frac{A-a}{2}m, y\right\rangle\right|^2 \leq C|A-a|^2, \tag{2.10}$$

and since

$$\left\|\frac{A+a}{2}y + \frac{A-a}{2}m\right\|^2 = \left|\frac{A+a}{2}\right|^2 + \left|\frac{A-a}{2}\right|^2$$

and

$$\left|\left\langle\frac{A+a}{2}y + \frac{A-a}{2}m, y\right\rangle\right|^2 = \left|\frac{A+a}{2}\right|^2$$

then, by (2.10), we obtain

$$\frac{|A-a|^2}{4} \leq C|A-a|^2,$$

which gives $C \geq \frac{1}{4}$, and the theorem is completely proved. ∎

2.3. **Applications for Isotonic Linear Functionals.** Let $F(T)$ be an algebra of real functions defined on T and L a subclass of $F(T)$ satisfying the conditions:

(i) $f, g \in L$ implies $f+g \in L$;
(ii) $f \in L$, $\alpha \in \mathbb{R}$ implies $\alpha f \in L$.

A functional A defined on L is an *isotonic linear functional* on L provided that

(a) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in L$;
(aa) $f \geq g$, that is, $f(t) \geq g(t)$ for all $t \in T$, implies $A(f) \geq A(g)$.

The functional A is *normalised* on L, provided that $\mathbf{1} \in L$, i.e., $\mathbf{1}(t) = 1$ for all $t \in T$, implies $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functionals are integrals, sums, etc.

Now, suppose that $h \in F(T)$, $h \geq 0$ is given and satisfies the properties that $fgh \in L$, $fh \in L$, $gh \in L$ for all $f, g \in L$. For a given isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$, define the mapping $(\cdot,\cdot)_{A,h} : L \times L \to \mathbb{R}$ by

$$(f, g)_{A,h} := \frac{A(fgh)}{A(h)}.$$

This functional satisfies the following properties:

(s) $(f, f)_{A,h} \geq 0$ for all $f \in L$;
(ss) $(\alpha f + \beta g, k)_{A,h} = \alpha (f, k)_{A,h} + \beta (g, k)_{A,h}$ for all $f, g, k \in L$ and $\alpha, \beta \in \mathbb{R}$;
(sss) $(f, g)_{A,h} = (g, f)_{A,h}$ for all $f, g \in L$.

The following reverse of Schwarz's inequality for positive linear functionals holds [5].

Proposition 1. *Let $f, g, h \in F(T)$ be such that $fgh \in L$, $f^2h \in L$, $g^2h \in L$. If m, M are real numbers such that*

$$mg \leq f \leq Mg \text{ on } F(T), \tag{2.11}$$

then for any isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$ we have the inequality

$$0 \leq A\left(hf^2\right) A\left(hg^2\right) - [A(hfg)]^2 \leq \frac{1}{4}(M-m)^2 A^2\left(hg^2\right). \tag{2.12}$$

The constant $\frac{1}{4}$ in (2.12) is sharp.

Proof. We observe that

$$(Mg - f, f - mg)_{A,h} = A[h(Mg-f)(f-mg)] \geq 0.$$

Applying Theorem 1 for $(\cdot,\cdot)_{A,h}$, we get

$$0 \leq (f,f)_{A,h}(g,g)_{A,h} - (f,g)_{A,h}^2 \leq \frac{1}{4}(M-m)^2 (g,g)_{A,h}^2,$$

which is clearly equivalent to (2.12). ∎

The following corollary holds.

Corollary 1. *Let $f, g \in F(T)$ such that fg, $f^2, g^2 \in F(T)$. If m, M are real numbers such that (2.11) holds, then*

$$0 \leq A\left(f^2\right) A\left(g^2\right) - A^2(fg) \leq \frac{1}{4}(M-m)^2 A^2\left(g^2\right). \tag{2.13}$$

The constant $\frac{1}{4}$ is sharp in (2.13).

Remark 1. *The condition (2.11) may be replaced with the weaker assumption*

$$(Mg-f,f-mg)_{A,h}\geq 0.$$

2.4. **Applications for Integrals.** Let (Ω,Σ,μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$.

Denote by $L_{\rho}^{2}(\Omega,\mathbb{K})$ the Hilbert space of all $\mathbb{K}$-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_{\Omega}\rho(t)\left|f(s)\right|^{2}d\mu(s)<\infty$, where $\rho:\Omega\to[0,\infty)$ is a measurable function on Ω.

The following proposition contains a reverse of the weighted Cauchy-Bunyakovsky-Schwarz's integral inequality [5].

Proposition 2. *Let $A,a\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $f,g\in L_{\rho}^{2}(\Omega,\mathbb{K})$. If*

$$\int_{\Omega}\operatorname{Re}\left[\left(Ag(s)-f(s)\right)\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]\rho(s)\,d\mu(s)\geq 0 \tag{2.14}$$

or equivalently,

$$\int_{\Omega}\rho(s)\left|f(s)-\frac{a+A}{2}g(s)\right|^{2}d\mu(s)\leq\frac{1}{4}\left|A-a\right|^{2}\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s),$$

holds, then one has the inequality

$$\begin{aligned}0&\leq\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)-\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}\\&\leq\frac{1}{4}\left|A-a\right|^{2}\left(\int_{\Omega}\rho(s)\left|g(s)\right|^{2}d\mu(s)\right)^{2}.\end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Follows by Theorem 1 applied for the inner product $\langle\cdot,\cdot\rangle_{\rho}:L_{\rho}^{2}(\Omega,\mathbb{K})\times L_{\rho}^{2}(\Omega,\mathbb{K})\to\mathbb{K}$,

$$\langle f,g\rangle_{\rho}:=\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s).$$

∎

Remark 2. *A sufficient condition for (2.14) to hold is*

$$\operatorname{Re}\left[\left(Ag(s)-f(s)\right)\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]\geq 0,\quad \textit{for}\quad \mu-a.e.\ s\in\Omega.$$

In the particular case $\rho=1$, we have the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

Corollary 2. *Let $a,A\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $f,g\in L^{2}(\Omega,\mathbb{K})$. If*

$$\int_{\Omega}\operatorname{Re}\left[\left(Ag(s)-f(s)\right)\left(\overline{f(s)}-\overline{a}\,\overline{g}(s)\right)\right]d\mu(s)\geq 0, \tag{2.15}$$

or equivalently,

$$\int_{\Omega}\left|f(s)-\frac{a+A}{2}g(s)\right|^{2}d\mu(s)\leq\frac{1}{4}\left|A-a\right|^{2}\int_{\Omega}\left|g(s)\right|^{2}d\mu(s),$$

holds, then one has the inequality

$$\begin{aligned}0 &\leq \int_{\Omega}|f(s)|^2 d\mu(s)\int_{\Omega}|g(s)|^2 d\mu(s) - \left|\int_{\Omega} f(s)\overline{g(s)}d\mu(s)\right|^2 \\ &\leq \frac{1}{4}|A-a|^2\left(\int_{\Omega}|g(s)|^2 d\mu(s)\right)^2.\end{aligned}$$

The constant $\frac{1}{4}$ *is best possible*

Remark 3. *If* $\mathbb{K}=\mathbb{R}$, *then a sufficient condition for either (2.14) or (2.15) to hold is*

$$ag(s) \leq f(s) \leq Ag(s), \quad \text{for } \mu - a.e.\ s \in \Omega,$$

where, in this case, $a, A \in \mathbb{R}$ *with* $A > a > 0$.

2.5. **Applications for Sequences.** For a given sequence $(w_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, consider the Hilbert space $\ell_w^2(\mathbb{K})$, $(\mathbb{K}=\mathbb{C},\mathbb{R})$, where

$$\ell_w^2(\mathbb{K}) := \left\{\overline{\mathbf{x}} = (x_i)_{i\in\mathbb{N}} \subset \mathbb{K}\left|\sum_{i=0}^{\infty} w_i|x_i|^2 < \infty\right.\right\}.$$

The following proposition that provides a reverse of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds.

Proposition 3. *Let* $a, A \in \mathbb{K}$ *and* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell_w^2(\mathbb{K})$. *If*

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[(Ay_i - x_i)(\overline{x_i} - \overline{a}\ \overline{y_i})\right] \geq 0, \tag{2.16}$$

then one has the inequality

$$0 \leq \sum_{i=0}^{\infty} w_i|x_i|^2\sum_{i=0}^{\infty} w_i|y_i|^2 - \left|\sum_{i=0}^{\infty} w_i x_i\overline{y_i}\right|^2 \leq \frac{1}{4}|A-a|^2\left(\sum_{i=0}^{\infty} w_i|y_i|^2\right)^2.$$

The constant $\frac{1}{4}$ *is sharp.*

Proof. Follows by Theorem 1 applied for the inner product $\langle\cdot,\cdot\rangle_w : \ell_w^2(\mathbb{K})\times\ell_w^2(\mathbb{K}) \to \mathbb{K}$,

$$\langle\overline{\mathbf{x}},\overline{\mathbf{y}}\rangle_w := \sum_{i=0}^{\infty} w_i x_i\overline{y_i}.$$

∎

Remark 4. *A sufficient condition for (2.16) to hold is*

$$\operatorname{Re}\left[(Ay_i - x_i)(\overline{x_i} - \overline{ay_i})\right] \geq 0, \quad \text{for all } i \in \mathbb{N}.$$

In the particular case $w_i = 1$, $i \in \mathbb{N}$, we have the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

Corollary 3. *Let* $a, A \in \mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *and* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell^2(\mathbb{K})$. *If*

$$\sum_{i=0}^{\infty} \operatorname{Re}\left[(Ay_i - x_i)(\overline{x_i} - \overline{ay_i})\right] \geq 0, \tag{2.17}$$

then one has the inequality

$$0\le\sum_{i=0}^{\infty}|x_i|^2\sum_{i=0}^{\infty}|y_i|^2-\left|\sum_{i=0}^{\infty}x_i\overline{y_i}\right|^2\le\frac{1}{4}|A-a|^2\left(\sum_{i=0}^{\infty}|y_i|^2\right)^2.$$

Remark 5. *If $\mathbb{K}=\mathbb{R}$, then a sufficient condition for either (2.16) or (2.17) to hold is*

$$ay_i\le x_i\le Ay_i \quad \text{for each} \quad i\in\mathbb{N},$$

with $A>a>0$.

3. A Generalisation of the Cassels and Greub-Reinboldt Inequalities

3.1. **Introduction.** The following result was proved by J.W.S. Cassels in 1951 (see Appendix 1 of [24]).

Theorem 2. *Let $\overline{\mathbf{a}}=(a_1,\dots,a_n)$, $\overline{\mathbf{b}}=(b_1,\dots,b_n)$ be sequences of positive real numbers and $\overline{\mathbf{w}}=(w_1,\dots,w_n)$ a sequence of nonnegative real numbers. Suppose that*

$$m=\min_{i=\overline{1,n}}\left\{\frac{a_i}{b_i}\right\} \quad \text{and} \quad M=\max_{i=\overline{1,n}}\left\{\frac{a_i}{b_i}\right\}. \tag{3.1}$$

Then one has the inequality

$$\frac{\sum_{i=1}^n w_ia_i^2\sum_{i=1}^n w_ib_i^2}{\left(\sum_{i=1}^n w_ia_ib_i\right)^2}\le\frac{(m+M)^2}{4mM}. \tag{3.2}$$

The equality holds in (3.2) when $w_1=\frac{1}{a_1b_1}$, $w_n=\frac{1}{a_nb_n}$, $w_2=\dots=w_{n-1}=0$, $m=\frac{a_n}{b_1}$ and $M=\frac{a_1}{b_n}$.

If one assumes that $0<a\le a_i\le A<\infty$ and $0<b\le b_i\le B<\infty$ for each $i\in\{1,\dots,n\}$, then by (3.2) we may obtain *Greub-Reinboldt's inequality* [12]

$$\frac{\sum_{i=1}^n w_ia_i^2\sum_{i=1}^n w_ib_i^2}{\left(\sum_{i=1}^n w_ia_ib_i\right)^2}\le\frac{(ab+AB)^2}{4abAB}.$$

The following "unweighted" Cassels' inequality also holds

$$\frac{\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_ib_i\right)^2}\le\frac{(m+M)^2}{4mM},$$

provided $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ satisfy (3.1). This inequality will produce the well known *Pólya-Szegö inequality* [20, pp. 57, 213-114], [17, pp. 71-72, 253-255]:

$$\frac{\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_ib_i\right)^2}\le\frac{(ab+AB)^2}{4abAB},$$

provided $0<a\le a_i\le A<\infty$ and $0<b\le b_i\le B<\infty$ for each $i\in\{1,\dots,n\}$.

In [18], C.P. Niculescu proved, amongst others, the following generalisation of Cassels' inequality:

Theorem 3. *Let E be a vector space endowed with a Hermitian product $\langle\cdot,\cdot\rangle$. Then*

$$\frac{\operatorname{Re}\langle x,y\rangle}{\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}}\ge\frac{2}{\sqrt{\frac{\omega}{\Omega}}+\sqrt{\frac{\Omega}{\omega}}}, \tag{3.3}$$

for every $x,y\in E$ and every $\omega,\Omega>0$ for which $\operatorname{Re}\langle x-\omega y,x-\Omega y\rangle\le0$.

In this section, by following [6], we establish a generalisation of (3.3) for complex numbers ω and Ω for which $\operatorname{Re}(\overline{\omega}\Omega) > 0$. Applications for isotonic linear functionals, integrals and sequences are also given.

3.2. **An Inequality in Real or Complex Inner Product Spaces.** The following reverse of Schwarz's inequality in inner product spaces holds [6].

Theorem 4. *Let $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ so that $\operatorname{Re}(\overline{a}A) > 0$. If $x, y \in H$ are such that*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{3.4}$$

then one has the inequality

$$\|x\| \|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} |\langle x, y\rangle| . \tag{3.5}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. We have, obviously, that

$$\begin{aligned} I &:= \operatorname{Re}\langle Ay - x, x - ay\rangle \\ &= \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right] - \|x\|^2 - \operatorname{Re}(\overline{a}A)\|y\|^2 \end{aligned}$$

and, thus, by (3.4), one has

$$\|x\|^2 + [\operatorname{Re}(\overline{a}A)] \cdot \|y\|^2 \leq \operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right],$$

which gives

$$\frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}} \|y\|^2 \leq \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}. \tag{3.6}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq,$$

valid for $p, q \geq 0$ and $\alpha > 0$, we deduce

$$2\|x\| \|y\| \leq \frac{1}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \|x\|^2 + [\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}} \|y\|^2 . \tag{3.7}$$

Utilizing (3.6) and (3.7) we deduce the first part of (3.5).

The second part is obvious by the fact that for $z \in \mathbb{C}$, $|\operatorname{Re}(z)| \leq |z|$.

Now, assume that the first inequality in (3.5) holds with a constant $c > 0$, i.e.,

$$\|x\| \|y\| \leq c \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}, \tag{3.8}$$

where a, A, x and y satisfy (3.5).

If we choose $a = A = 1$, $y = x \neq 0$, then obviously (3.4) holds and from (3.8) we obtain

$$\|x\|^2 \leq 2c\|x\|^2,$$

giving $c \geq \frac{1}{2}$.

The theorem is completely proved. ∎

The following corollary is a natural consequence of the above theorem [6].

Corollary 4. *Let $m, M > 0$. If $x, y \in H$ are such that*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0,$$

then one has the inequality

$$\|x\|\,\|y\| \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} |\langle x, y\rangle| . \tag{3.9}$$

The constant $\frac{1}{2}$ is sharp in (3.9).

Remark 6. *The inequality (3.9) is equivalent to Niculescu's inequality (3.3).*

The following corollary is also obvious [6].

Corollary 5. *With the assumptions of Corollary 4, we have*

$$\begin{aligned} 0 &\leq \|x\|\,\|y\| - |\langle x, y\rangle| \leq \|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} |\langle x, y\rangle| \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} 0 &\leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 \\ &\leq \frac{(M-m)^2}{4mM} [\operatorname{Re}\langle x, y\rangle]^2 \leq \frac{(M-m)^2}{4mM} |\langle x, y\rangle|^2 . \end{aligned} \tag{3.11}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are sharp.

Proof. If we subtract $\operatorname{Re}\langle x, y\rangle \geq 0$ from the first inequality in (3.9), we get

$$\begin{aligned} \|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle &\leq \left(\frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} - 1\right) \operatorname{Re}\langle x, y\rangle \\ &= \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \end{aligned}$$

which proves the third inequality in (3.10). The other ones are obvious.

Now, if we square the first inequality in (3.9) and then subtract $[\operatorname{Re}\langle x, y\rangle]^2$, we get

$$\begin{aligned} \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 &\leq \left[\frac{(M+m)^2}{4mM} - 1\right] [\operatorname{Re}\langle x, y\rangle]^2 \\ &= \frac{(M-m)^2}{4mM} [\operatorname{Re}\langle x, y\rangle]^2 \end{aligned}$$

which proves the third inequality in (3.11). The other ones are obvious. ∎

3.3. Applications for Isotonic Linear Functionals.

The following proposition holds [6].

Proposition 4. *Let $f, g, h \in F(T)$ be such that $fgh \in L$, $f^2 h \in L$, $g^2 h \in L$. If $m, M > 0$ are such that*

$$mg \leq f \leq Mg \text{ on } F(T), \tag{3.12}$$

then for any isotonic linear functional $A : L \to \mathbb{R}$ with $A(h) > 0$, we have the inequality

$$1 \le \frac{A\left(f^2 h\right) A\left(g^2 h\right)}{A^2 (fgh)} \le \frac{(M+m)^2}{4mM}. \tag{3.13}$$

The constant $\frac{1}{4}$ in (3.13) is sharp.

Proof. We observe that

$$(Mg - f, f - mg)_{A,h} = A\left[h\left(Mg - f\right)\left(f - mg\right)\right] \ge 0.$$

Applying Corollary 4 for $(\cdot, \cdot)_{A,h}$ we get

$$1 \le \frac{(f,f)_{A,h} (g,g)_{A,h}}{(f,g)_{A,h}^2} \le \frac{(M+m)^2}{4mM},$$

which is clearly equivalent to (3.13). ∎

The following additive versions of (3.13) also hold [6].

Corollary 6. *With the assumption in Proposition 4, one has*

$$\begin{aligned} 0 &\le \left[A\left(f^2 h\right) A\left(g^2 h\right)\right]^{\frac{1}{2}} - A(hfg) \\ &\le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} A(hfg) \end{aligned}$$

and

$$\begin{aligned} 0 &\le A\left(f^2 h\right) A\left(g^2 h\right) - A^2 (fgh) \\ &\le \frac{(M-m)^2}{4mM} A^2 (fgh). \end{aligned}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are sharp.

Remark 7. *The condition (3.12) may be replaced with the weaker assumption*

$$(Mg - f, f - mg)_{A,h} \ge 0. \tag{3.14}$$

Remark 8. *With the assumption (3.12) or (3.14) and if $f, g \in F(T)$ with $fg, f^2, g^2 \in L$, then one has the inequalities*

$$1 \le \frac{A\left(f^2\right) A\left(g^2\right)}{A^2 (fg)} \le \frac{(M+m)^2}{4mM},$$

$$\begin{aligned} 0 &\le \left[A\left(f^2\right) A\left(g^2\right)\right]^{\frac{1}{2}} - A(fg) \\ &\le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} A(fg) \end{aligned}$$

and

$$0 \le A\left(f^2\right) A\left(g^2\right) - A^2 (fg) \le \frac{(M-m)^2}{4mM} A^2 (fg).$$

3.4. **Applications for Integrals.** The following proposition contains a reverse of the weighted Cauchy-Bunyakovsky-Schwarz integral inequality.

Proposition 5. *Let* $A, a \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *with* $\operatorname{Re}(\overline{a}A) > 0$ *and* $f, g \in L^2_\rho(\Omega, \mathbb{K})$. *If*

$$\int_\Omega \operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\;\overline{g}(s)\right)\right]\rho(s)\,d\mu(s) \geq 0, \tag{3.15}$$

then one has the inequality

$$\begin{aligned}\left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} &\leq \frac{1}{2}\cdot\frac{\int_\Omega \rho(s)\operatorname{Re}\left[A\overline{f(s)}g(s) + \overline{a}f(s)\overline{g(s)}\right]d\mu(s)}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \\ &\leq \frac{1}{2}\cdot\frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}\left|\int_\Omega \rho(s)f(s)\overline{g(s)}d\mu(s)\right|.\end{aligned} \tag{3.16}$$

The constant $\frac{1}{2}$ *is sharp in (3.16).*

Proof. Follows by Theorem 4 applied for the inner product $\langle\cdot,\cdot\rangle_\rho : L^2_\rho(\Omega, \mathbb{K}) \times L^2_\rho(\Omega, \mathbb{K}) \to \mathbb{K}$,

$$\langle f, g\rangle := \int_\Omega \rho(s)f(s)\overline{g(s)}d\mu(s).$$

∎

Remark 9. *A sufficient condition for (3.15) to hold is*

$$\operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\overline{g(s)}\right)\right] \geq 0, \quad \text{for } \mu\text{-a.e. } s \in \Omega.$$

In the particular case $\rho = 1$, we have the following result.

Corollary 7. *Let* $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *with* $\operatorname{Re}(\overline{a}A) > 0$ *and* $f, g \in L^2(\Omega, \mathbb{K})$. *If*

$$\int_\Omega \operatorname{Re}\left[(Ag(s) - f(s))\left(\overline{f(s)} - \overline{a}\overline{g(s)}\right)\right]d\mu(s) \geq 0, \tag{3.17}$$

then one has the inequality

$$\begin{aligned}\left[\int_\Omega |f(s)|^2 d\mu(s)\int_\Omega |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} &\leq \frac{1}{2}\cdot\frac{\int_\Omega \operatorname{Re}\left[A\overline{f(s)}g(s) + \overline{a}f(s)\overline{g(s)}\right]d\mu(s)}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}} \\ &\leq \frac{1}{2}\cdot\frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{\frac{1}{2}}}\left|\int_\Omega f(s)\overline{g(s)}d\mu(s)\right|.\end{aligned}$$

Remark 10. *If* $\mathbb{K} = \mathbb{R}$, *then a sufficient condition for either (3.15) or (3.17) to hold is*

$$ag(s) \leq f(s) \leq Ag(s), \quad \text{for } \mu\text{-a.e. } s \in \Omega,$$

where, in this case, $a, A \in \mathbb{R}$ *with* $A > a > 0$.

If a, A are real positive constants, then the following proposition holds.

Proposition 6. *Let $m, M > 0$. If $f, g \in L_{\rho}^{2}(\Omega, \mathbb{K})$ are such that*

$$\int_{\Omega} \rho(s) \operatorname{Re}\left[(Mg(s) - f(s))\left(\overline{f(s)} - m\overline{g}(s)\right)\right] d\mu(s) \geq 0,$$

then one has the inequality

$$\left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \int_{\Omega} \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s).$$

The proof follows by Corollary 4 applied for the inner product

$$\langle f, g\rangle_{\rho} := \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s).$$

The following additive versions also hold [6].

Corollary 8. *With the assumptions in Proposition 6, one has the inequalities*

$$\begin{aligned} 0 &\leq \left[\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} \\ &\qquad - \int_{\Omega} \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \int_{\Omega} \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\ &\qquad - \left(\int_{\Omega} \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s)\right)^2 \\ &\leq \frac{(M-m)^2}{4mM} \left(\int_{\Omega} \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s)\right)^2. \end{aligned}$$

Remark 11. *If $\mathbb{K} = \mathbb{R}$, a sufficient condition for (3.15) to hold is*

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu\text{-a.e. } s \in \Omega,$$

where $M > m > 0$.

3.5. **Applications for Sequences.** For a given sequence $(w_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, consider the Hilbert space $\ell_w^2(\mathbb{K})$, $(\mathbb{K} = \mathbb{C}, \mathbb{R})$, where

$$\ell_w^2(\mathbb{K}) := \left\{ \overline{\mathbf{x}} = (x_i)_{i\in\mathbb{N}} \subset \mathbb{K} \,\middle|\, \sum_{i=0}^{\infty} w_i |x_i|^2 < \infty \right\}.$$

The following proposition that provides a reverse of the weighted Cauchy-Bunyakovsky-Schwarz inequality for complex numbers holds [6].

Proposition 7. *Let $a, A \in \mathbb{K}$ with $\operatorname{Re}(\overline{a}A) > 0$ and $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell_w^2(\mathbb{K})$. If*

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[(Ay_i - x_i)(\overline{x_i} - \overline{ay_i})\right] \geq 0, \tag{3.18}$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[A\overline{x_i}y_i + \overline{a}x_i\overline{y_i}\right]}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=0}^{\infty} w_i x_i \overline{y_i}\right|. \tag{3.19}$$

The constant $\frac{1}{2}$ *is sharp in (3.19).*

Proof. Follows by Theorem 4 applied for the inner product $\langle \cdot, \cdot \rangle : \ell_w^2(\mathbb{K}) \times \ell_w^2(\mathbb{K}) \to \mathbb{K}$,

$$\langle \overline{\mathbf{x}}, \overline{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i \overline{y_i}.$$

∎

Remark 12. *A sufficient condition for (3.18) to hold is*

$$\operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\overline{x_i} - \overline{ay_i}\right)\right] \geq 0 \quad \text{for all } i \in \mathbb{N}. \tag{3.20}$$

In the particular case $\rho = 1$, we have the following result.

Corollary 9. *Let* $a, A \in \mathbb{K}$ *with* $\operatorname{Re}(\overline{a}A) > 0$ *and* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell^2(\mathbb{K})$. *If*

$$\sum_{i=0}^{\infty} \operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\overline{x_i} - \overline{ay_i}\right)\right] \geq 0,$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} |x_i|^2 \sum_{i=0}^{\infty} |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=0}^{\infty} \operatorname{Re}\left[A\overline{x_i}y_i + \overline{a}x_i\overline{y_i}\right]}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\overline{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=0}^{\infty} x_i \overline{y_i}\right|.$$

Remark 13. *If* $\mathbb{K} = \mathbb{R}$, *then a sufficient condition for either (3.18) or (3.20) to hold is*

$$ay_i \leq x_i \leq Ay_i \quad \text{for each} \quad i \in \{1, \dots, n\},$$

where, in this case, $a, A \in \mathbb{R}$ *with* $A > a > 0$.

For $a = m$, $A = M$, then the following proposition also holds.

Proposition 8. *Let* $m, M > 0$. *If* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in \ell_w^2(\mathbb{K})$ *such that*

$$\sum_{i=0}^{\infty} w_i \operatorname{Re}\left[\left(My_i - x_i\right)\left(\overline{x_i} - m\overline{y_i}\right)\right] \geq 0, \tag{3.21}$$

then one has the inequality

$$\left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{M + m}{\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re}\left(x_i \overline{y_i}\right).$$

The proof follows by Corollary 4 applied for the inner product

$$\langle \overline{\mathbf{x}}, \overline{\mathbf{y}} \rangle_w := \sum_{i=0}^{\infty} w_i x_i, \overline{y_i}.$$

The following additive version also holds [6].

Corollary 10. *With the assumptions in Proposition 8, one has the inequalities*

$$0 \leq \left[\sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 \right]^{\frac{1}{2}} - \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i})$$
$$\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i})$$

and

$$0 \leq \sum_{i=0}^{\infty} w_i |x_i|^2 \sum_{i=0}^{\infty} w_i |y_i|^2 - \left[\sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \right]^2$$
$$\leq \frac{(M-m)^2}{4mM} \left[\sum_{i=0}^{\infty} w_i \operatorname{Re}(x_i \overline{y_i}) \right]^2 .$$

Remark 14. *If $\mathbb{K} = \mathbb{R}$, a sufficient condition for (3.21) to hold is*

$$my_i \leq x_i \leq My_i \quad \text{for each} \quad i \in \mathbb{N},$$

where $M > m > 0$.

4. Quadratic Reverses of Schwarz's Inequality

4.1. Two Better Reverse Inequalities.

It has been proven in [8], that

$$0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\phi - \varphi|^2 - \left| \frac{\phi + \varphi}{2} - \langle x, e \rangle \right|^2, \tag{4.1}$$

provided, either

$$\operatorname{Re} \langle \phi e - x, x - \varphi e \rangle \geq 0, \tag{4.2}$$

or equivalently,

$$\left\| x - \frac{\phi + \varphi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| , \tag{4.3}$$

holds, where $e \in H$, $\|e\| = 1$. The constant $\frac{1}{4}$ in (4.1) is best possible.

If we choose $e = \frac{y}{\|y\|}$, $\phi = \Gamma \|y\|$, $\varphi = \gamma \|y\|$ $(y \neq 0)$, $\Gamma, \gamma \in \mathbb{K}$, then by (4.2) and (4.3) we have,

$$\operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0, \tag{4.4}$$

or equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\| , \tag{4.5}$$

implying the following reverse of Schwarz's inequality:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^4 - \left| \frac{\Gamma + \gamma}{2} \|y\|^2 - \langle x, y \rangle \right|^2 . \tag{4.6}$$

The constant $\frac{1}{4}$ in (4.6) is sharp.

Note that, this inequality is an improvement of (2.6), but it may not be very convenient for applications.

In [3], it has also been proven that

$$0 \le \|x\|^2 - |\langle x, e\rangle|^2 \le \frac{1}{4}|\phi - \varphi|^2 - \operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle, \tag{4.7}$$

provided either (4.2) or (4.3) holds true.

If we make the same choice for e, Φ and φ as above, then we deduce the inequality

$$0 \le \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \le \frac{1}{4}|\Gamma - \gamma|^2 \|y\|^4 - \|y\|^2 \operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \tag{4.8}$$

provided either (4.4) or (4.5) holds true.

The constant $\frac{1}{4}$ is best possible in (4.8).

We note that (4.8) is another improvement of (2.6). Moreover, the bounds provided by (4.6) and (4.8) cannot be compared in general, meaning that one may provide better bounds than the other for different choices of the parameters involved.

4.2. **A Reverse of Schwarz's Inequality Under More General Assumptions.**

The following result holds [9].

Theorem 5. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ $(\mathbb{K} = \mathbb{R},\ \mathbb{K} = \mathbb{C})$ and $x, a \in H$, $r > 0$ are such that*

$$x \in \overline{B}(a, r) := \{z \in H | \|z - a\| \le r\}.$$

(i) *If $\|a\| > r$, then we have the inequalities*

$$0 \le \|x\|^2 \|a\|^2 - |\langle x, a\rangle|^2 \le \|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 \le r^2 \|x\|^2. \tag{4.9}$$

The constant $C = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller one.

(ii) *If $\|a\| = r$, then*

$$\|x\|^2 \le 2 \operatorname{Re}\langle x, a\rangle \le 2|\langle x, a\rangle|. \tag{4.10}$$

The constant 2 is best possible in both inequalities.

(iii) *If $\|a\| < r$, then*

$$\|x\|^2 \le r^2 - \|a\|^2 + 2\operatorname{Re}\langle x, a\rangle \le r^2 - \|a\|^2 + 2|\langle x, a\rangle|. \tag{4.11}$$

Here the constant 2 is also best possible.

Proof. Since $x \in \overline{B}(a, r)$, then obviously $\|x - a\|^2 \le r^2$, which is equivalent to

$$\|x\|^2 + \|a\|^2 - r^2 \le 2\operatorname{Re}\langle x, a\rangle. \tag{4.12}$$

(i) If $\|a\| > r$, then we may divide (4.12) by $\sqrt{\|a\|^2 - r^2} > 0$ getting

$$\frac{\|x\|^2}{\sqrt{\|a\|^2 - r^2}} + \sqrt{\|a\|^2 - r^2} \le \frac{2\operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}}. \tag{4.13}$$

Using the elementary inequality

$$\alpha p + \frac{1}{\alpha} q \ge 2\sqrt{pq}, \quad \alpha > 0, \quad p, q \ge 0,$$

we may state that

$$2\|x\| \leq \frac{\|x\|^2}{\sqrt{\|a\|^2 - r^2}} + \sqrt{\|a\|^2 - r^2}. \tag{4.14}$$

Making use of (4.13) and (4.14), we deduce

$$\|x\| \sqrt{\|a\|^2 - r^2} \leq \operatorname{Re}\langle x, a\rangle, \tag{4.15}$$

which is an interesting inequality in itself as well.

Taking the square in (4.15) and re-arranging the terms, we deduce the third inequality in (4.9). The others are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that, there exists a constant $c > 0$ such that

$$\|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 \leq cr^2 \|x\|^2, \tag{4.16}$$

provided $x \in \overline{B}(a, r)$ and $\|a\| > r$.

Let $r = \sqrt{\varepsilon} > 0$, $\varepsilon \in (0, 1)$, $a, e \in H$ with $\|a\| = \|e\| = 1$ and $a \perp e$. Put $x = a + \sqrt{\varepsilon} e$. Then obviously $x \in \overline{B}(a, r)$, $\|a\| > r$ and $\|x\|^2 = \|a\|^2 + \varepsilon \|e\|^2 = 1 + \varepsilon$, $\operatorname{Re}\langle x, a\rangle = \|a\|^2 = 1$, and thus $\|x\|^2 \|a\|^2 - [\operatorname{Re}\langle x, a\rangle]^2 = \varepsilon$. Using (4.16), we may write that

$$\varepsilon \leq c\varepsilon (1 + \varepsilon), \quad \varepsilon > 0$$

giving

$$c + c\varepsilon \geq 1 \text{ for any } \varepsilon > 0. \tag{4.17}$$

Letting $\varepsilon \to 0+$, we get from (4.17) that $c \geq 1$, and the sharpness of the constant is proved.

(ii) The inequality (4.10) is obvious by (4.12) since $\|a\| = r$. The best constant follows in a similar way to the above.

(iii) The inequality (4.11) is obvious. The best constant may be proved in a similar way to the above. We omit the details.

∎

The following reverse of Schwarz's inequality holds [9].

Theorem 6. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $x, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ such that either*

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{4.18}$$

or equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|, \tag{4.19}$$

holds.

(i) *If $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$, then we have the inequalities*

$$\begin{aligned} \|x\|^2 \|y\|^2 &\leq \frac{1}{4} \cdot \frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma \overline{\gamma})} \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma \overline{\gamma})} |\langle x, y\rangle|^2. \end{aligned} \tag{4.20}$$

The constant $\frac{1}{4}$ *is best possible in both inequalities.*

(ii) *If* $\operatorname{Re}(\Gamma\overline{\gamma}) = 0$, *then*

$$\|x\|^2 \le \operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right] \le |\Gamma + \gamma|\, |\langle x, y\rangle| .$$

(iii) *If* $\operatorname{Re}(\Gamma\overline{\gamma}) < 0$, *then*

$$\begin{aligned}\|x\|^2 &\le -\operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 + \operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right] \\ &\le -\operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 + |\Gamma + \gamma|\, |\langle x, y\rangle| .\end{aligned}$$

Proof. The proof of the equivalence between the inequalities (4.18) and (4.19) follows by the fact that in an inner product space $\operatorname{Re}\langle Z - x, x - z\rangle \ge 0$ for $x, z, Z \in H$ is equivalent with $\left\|x - \frac{z+Z}{2}\right\| \le \frac{1}{2}\|Z - z\|$ (see for example [7]).

Consider, for $y \ne 0$, $a = \frac{\gamma+\Gamma}{2}y$ and $r = \frac{1}{2}|\Gamma - \gamma|\,\|y\|$. Then

$$\|a\|^2 - r^2 = \frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4}\|y\|^2 = \operatorname{Re}(\Gamma\overline{\gamma})\|y\|^2 .$$

(i) If $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then the hypothesis of (i) in Theorem 5 is satisfied, and by the second inequality in (4.9) we have

$$\|x\|^2 \frac{|\Gamma + \gamma|^2}{4}\|y\|^2 - \frac{1}{4}\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2 \le \frac{1}{4}|\Gamma - \gamma|^2\|x\|^2\|y\|^2$$

from where we derive

$$\frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4}\|x\|^2\|y\|^2 \le \frac{1}{4}\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2 ,$$

giving the first inequality in (4.20).

The second inequality is obvious.

To prove the sharpness of the constant $\frac{1}{4}$, assume that the first inequality in (4.20) holds with a constant $c > 0$, i.e.,

$$\|x\|^2\|y\|^2 \le c \cdot \frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma} + \overline{\gamma}\right)\langle x, y\rangle\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})}, \tag{4.21}$$

provided $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$ and either (4.18) or (4.19) holds.

Assume that $\Gamma, \gamma > 0$, and let $x = \gamma y$. Then (4.18) holds and by (4.21) we deduce

$$\gamma^2\|y\|^4 \le c \cdot \frac{(\Gamma + \gamma)^2\gamma^2\|y\|^4}{\Gamma\gamma}$$

giving

$$\Gamma\gamma \le c(\Gamma + \gamma)^2 \quad \text{for any} \quad \Gamma, \gamma > 0. \tag{4.22}$$

Let $\varepsilon \in (0, 1)$ and choose in (4.22), $\Gamma = 1+\varepsilon$, $\gamma = 1-\varepsilon > 0$ to get $1-\varepsilon^2 \le 4c$ for any $\varepsilon \in (0, 1)$. Letting $\varepsilon \to 0+$, we deduce $c \ge \frac{1}{4}$, and the sharpness of the constant is proved.

(ii) and (iii) are obvious and we omit the details.

∎

Remark 15. *We observe that the second bound in (4.20) for* $\|x\|^2\|y\|^2$ *is better than the second bound provided by (3.5).*

The following corollary provides a reverse inequality for the additive version of Schwarz's inequality [9].

Corollary 11. *With the assumptions of Theorem 6 and if* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, *then we have the inequality:*

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} |\langle x, y\rangle|^2 . \tag{4.23}$$

The constant $\frac{1}{4}$ *is best possible in (4.23).*

The proof is obvious from (4.20) on subtracting in both sides the same quantity $|\langle x, y\rangle|^2$. The sharpness of the constant may be proven in a similar manner to the one incorporated in the proof of (i), Theorem 6. We omit the details.

For other recent results in connection to Schwarz's inequality, see [22], [11] and [13].

4.3. **Reverses of the Triangle Inequality.** The following reverse of the triangle inequality holds [9].

Proposition 9. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $x, a \in H$, $r > 0$ *are such that*

$$\|x - a\| \leq r < \|a\| .$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \|x\| + \|a\| - \|x + a\| \\ &\leq \sqrt{2} r \cdot \sqrt{\frac{\operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)}}. \end{aligned} \tag{4.24}$$

Proof. Using the inequality (4.15), we may write that

$$\|x\| \|a\| \leq \frac{\|a\| \operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}},$$

which gives

$$\begin{aligned} 0 &\leq \|x\| \|a\| - \operatorname{Re}\langle x, a\rangle \\ &\leq \frac{\|a\| - \sqrt{\|a\|^2 - r^2}}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re}\langle x, a\rangle \\ &= \frac{r^2 \operatorname{Re}\langle x, a\rangle}{\sqrt{\|a\|^2 - r^2}\left(\sqrt{\|a\|^2 - r^2} + \|a\|\right)} \end{aligned} \tag{4.25}$$

which is an interesting inequality in itself as well.

Since

$$(\|x\| + \|a\|)^2 - \|x + a\|^2 = 2\left(\|x\| \|a\| - \operatorname{Re}\langle x, a\rangle\right),$$

then by (4.25), we have

$$\|x\|+\|a\| \leq \sqrt{\|x+a\|^{2}+\frac{2r^{2}\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^{2}-r^{2}}\left(\sqrt{\|a\|^{2}-r^{2}}+\|a\|\right)}}$$
$$\leq \|x+a\|+\sqrt{2}r\cdot\sqrt{\frac{\operatorname{Re}\langle x,a\rangle}{\sqrt{\|a\|^{2}-r^{2}}\left(\sqrt{\|a\|^{2}-r^{2}}+\|a\|\right)}},$$

giving the desired inequality (4.24). ∎

The following proposition providing a simpler reverse for the triangle inequality also holds [9].

Proposition 10. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y\in H$, $M>m>0$ such that either*

$$\operatorname{Re}\langle My-x,x-my\rangle\geq 0,$$

or equivalently,

$$\left\|x-\frac{M+m}{2}\cdot y\right\|\leq\frac{1}{2}(M-m)\|y\|,$$

holds. Then we have the inequality

$$0\leq\|x\|+\|y\|-\|x+y\|\leq\frac{\sqrt{M}-\sqrt{m}}{\sqrt[4]{mM}}\sqrt{\operatorname{Re}\langle x,y\rangle}. \tag{4.26}$$

Proof. Choosing in (4.15), $a=\frac{M+m}{2}y$, $r=\frac{1}{2}(M-m)\|y\|$ we get

$$\|x\|\|y\|\sqrt{Mm}\leq\frac{M+m}{2}\operatorname{Re}\langle x,y\rangle,$$

giving

$$0\leq\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{2\sqrt{mM}}\operatorname{Re}\langle x,y\rangle.$$

Following the same argument as in the proof of Proposition 9, we deduce the desired inequality (4.26). ∎

For some results related to triangle inequality in inner product spaces, see [2], [14], [16] and [21].

4.4. **Integral Inequalities.** Let (Ω,Σ,μ) be a measurable space consisting of a set Ω, Σ a $\sigma-$algebra of parts and μ a countably additive and positive measure on Σ with values in $\mathbb{R}\cup\{\infty\}$. Let $\rho\geq 0$ be a $g-$measurable function on Ω with $\int_{\Omega}\rho(s)\,d\mu(s)=1$. Denote by $L_{\rho}^{2}(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)<\infty.$$

It is obvious that the following inner product

$$\langle f,g\rangle_{\rho}:=\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s),$$

generates the norm $\|f\|_\rho := \left(\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\right)^{\frac{1}{2}}$ of $L^2_\rho(\Omega,\mathbb{K})$, and all the above results may be stated for integrals.

It is important to observe that, if

$$\operatorname{Re}\left[f(s)\overline{g(s)}\right] \geq 0, \quad \text{for } \mu - \text{a.e. } s \in \Omega,$$

then, obviously,

$$\operatorname{Re}\langle f,g\rangle_\rho = \operatorname{Re}\left[\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right] = \int_\Omega \rho(s)\operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) \geq 0. \tag{4.27}$$

The reverse is evidently not true in general.

Moreover, if the space is real, i.e., $\mathbb{K}=\mathbb{R}$, then a sufficient condition for (4.27) to hold is:

$$f(s) \geq 0, \quad g(s) \geq 0, \quad \text{for } \mu - \text{a.e. } s \in \Omega.$$

We now provide, by the use of certain results obtained above, some integral inequalities that may be used in practical applications.

Proposition 11. *Let $f,g \in L^2_\rho(\Omega,\mathbb{K})$ and $r>0$ with the properties that*

$$|f(s)-g(s)| \leq r \leq |g(s)|, \quad \textit{for } \mu - \textit{a.e. } s \in \Omega. \tag{4.28}$$

Then we have the inequalities

$$\begin{aligned} 0 &\leq \int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s) \\ &\qquad - \left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right|^2 \\ &\leq \int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s) \\ &\qquad - \left[\int_\Omega \rho(s)\operatorname{Re}\left(f(s)\overline{g(s)}\right) d\mu(s)\right]^2 \\ &\leq r^2 \int_\Omega \rho(s)|g(s)|^2 d\mu(s). \end{aligned} \tag{4.29}$$

The constant $c=1$ in front of r^2 is best possible.

The proof follows by Theorem 5 and we omit the details.

The following results also holds [9].

Proposition 12. *Let $f,g \in L^2_\rho(\Omega,\mathbb{K})$ and $\gamma,\Gamma \in \mathbb{K}$ such that* $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$ *and*

$$\operatorname{Re}\left[(\Gamma g(s) - f(s))\left(\overline{f(s)} - \overline{\gamma}\overline{g(s)}\right)\right] \geq 0, \quad \textit{for } \mu - \textit{a.e. } s \in \Omega.$$

Then we have the inequalities

$$\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s) \leq \frac{1}{4}\cdot\frac{\left\{\operatorname{Re}\left[\left(\overline{\Gamma}+\overline{\gamma}\right)\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right]\right\}^{2}}{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)} \leq \frac{1}{4}\cdot\frac{|\Gamma+\gamma|^{2}}{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}. \tag{4.30}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof follows by Theorem 6 and we omit the details.

Corollary 12. *With the assumptions of Proposition 12, we have the inequality*

$$0\leq\int_{\Omega}\rho(s)|f(s)|^{2}d\mu(s)\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)-\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2} \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^{2}}{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|^{2}. \tag{4.31}$$

The constant $\frac{1}{4}$ is best possible.

Remark 16. *If the space is real and we assume, for $M>m>0$, that*

$$mg(s)\leq f(s)\leq Mg(s)\,,\quad \text{for } \mu-\text{a.e. } s\in\Omega,$$

then, by (4.30) and (4.31), we deduce the inequalities

$$\int_{\Omega}\rho(s)[f(s)]^{2}d\mu(s)\int_{\Omega}\rho(s)[g(s)]^{2}d\mu(s) \leq\frac{1}{4}\cdot\frac{(M+m)^{2}}{mM}\left[\int_{\Omega}\rho(s)f(s)g(s)d\mu(s)\right]^{2} \tag{4.32}$$

and

$$0\leq\int_{\Omega}\rho(s)[f(s)]^{2}d\mu(s)\int_{\Omega}\rho(s)[g(s)]^{2}d\mu(s)-\left[\int_{\Omega}\rho(s)f(s)g(s)d\mu(s)\right]^{2} \leq\frac{1}{4}\cdot\frac{(M-m)^{2}}{mM}\left[\int_{\Omega}\rho(s)f(s)g(s)d\mu(s)\right]^{2}. \tag{4.33}$$

The inequality (4.32) is known in the literature as Cassel's inequality.

5. More Reverses of Schwarz's Inequality

5.1. **General Results.** The following result holds [10].

Theorem 7. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x,a\in H$ and $r>0$. If*

$$x\in\bar{B}(a,r):=\{z\in H|\,\|z-a\|\leq r\}\,, \tag{5.1}$$

then we have the inequalities:

$$0 \le \|x\|\,\|a\| - |\langle x, a\rangle| \le \|x\|\,\|a\| - |\mathrm{Re}\,\langle x, a\rangle| \le \|x\|\,\|a\| - \mathrm{Re}\,\langle x, a\rangle \le \frac{1}{2}r^2. \tag{5.2}$$

The constant $\frac{1}{2}$ *is best possible in (5.2) in the sense that it cannot be replaced by a smaller constant.*

Proof. The condition (5.1) is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \le 2\,\mathrm{Re}\,\langle x, a\rangle + r^2. \tag{5.3}$$

Using the elementary inequality

$$2\,\|x\|\,\|a\| \le \|x\|^2 + \|a\|^2, \quad a, x \in H$$

and (5.3), we deduce

$$2\,\|x\|\,\|a\| \le 2\,\mathrm{Re}\,\langle x, a\rangle + r^2,$$

giving the last inequality in (5.2). The other inequalities are obvious.

To prove the sharpness of the constant $\frac{1}{2}$, assume that

$$0 \le \|x\|\,\|a\| - \mathrm{Re}\,\langle x, a\rangle \le cr^2 \tag{5.4}$$

for any $x, a \in H$ and $r > 0$ satisfying (5.1).

Assume that $a, e \in H$, $\|a\| = \|e\| = 1$ and $e \perp a$. If $r = \sqrt{\varepsilon}$, $\varepsilon > 0$ and if we define $x = a + \sqrt{\varepsilon}e$, then $\|x - a\| = \sqrt{\varepsilon} = r$ showing that the condition (5.1) is fulfilled.

On the other hand,

$$\begin{aligned}\|x\|\,\|a\| - \mathrm{Re}\,\langle x, a\rangle &= \sqrt{\left\|a + \sqrt{\varepsilon}e\right\|^2} - \mathrm{Re}\,\left\langle a + \sqrt{\varepsilon}e, a\right\rangle \\ &= \sqrt{\|a\|^2 + \varepsilon\,\|e\|^2} - \|a\|^2 \\ &= \sqrt{1+\varepsilon} - 1.\end{aligned}$$

Utilising (5.4), we conclude that

$$\sqrt{1+\varepsilon} - 1 \le c\varepsilon \quad \text{for any } \varepsilon > 0. \tag{5.5}$$

Multiplying (5.5) by $\sqrt{1+\varepsilon} + 1 > 0$ and then dividing by $\varepsilon > 0$, we get

$$\left(\sqrt{1+\varepsilon} + 1\right) c \ge 1 \quad \text{for any} \quad \varepsilon > 0. \tag{5.6}$$

Letting $\varepsilon \to 0+$ in (5.6), we deduce $c \ge \frac{1}{2}$, and the theorem is proved. ∎

The following result also holds [10].

Theorem 8. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ *and* $x, y \in H$, $\gamma, \Gamma \in \mathbb{K}$ $(\Gamma \ne -\gamma)$ *so that either*

$$\mathrm{Re}\,\langle \Gamma y - x, x - \gamma y\rangle \ge 0, \tag{5.7}$$

or equivalently,

$$\left\|x - \frac{\gamma+\Gamma}{2}y\right\| \le \frac{1}{2}\,|\Gamma - \gamma|\,\|y\|, \tag{5.8}$$

holds. Then we have the inequalities

$$0 \leq \|x\| \|y\| - |\langle x, y\rangle| \leq \|x\| \|y\| - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right]\right| \leq \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right] \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2. \tag{5.9}$$

The constant $\frac{1}{4}$ in the last inequality is best possible.

Proof. Consider for $a, y \neq 0$, $a = \frac{\Gamma+\gamma}{2}\cdot y$ and $r = \frac{1}{2}|\Gamma-\gamma| \|y\|$. Thus from (5.2), we get

$$\begin{aligned} 0 &\leq \|x\| \left|\frac{\Gamma+\gamma}{2}\right| \|y\| - \left|\frac{\Gamma+\gamma}{2}\right| |\langle x, y\rangle| \\ &\leq \|x\| \left|\frac{\Gamma+\gamma}{2}\right| \|y\| - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{2}\langle x, y\rangle\right]\right| \\ &\leq \|x\| \left|\frac{\Gamma+\gamma}{2}\right| \|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{2}\langle x, y\rangle\right] \\ &\leq \frac{1}{8}\cdot|\Gamma-\gamma|^2 \|y\|^2. \end{aligned}$$

Dividing by $\frac{1}{2}|\Gamma+\gamma| > 0$, we deduce the desired inequality (5.9).

To prove the sharpness of the constant $\frac{1}{4}$, assume that there exists a $c > 0$ such that:

$$\|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\langle x, y\rangle\right] \leq c\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2, \tag{5.10}$$

provided either (5.7) or (5.8) holds.

Consider the real inner product space $\left(\mathbb{R}^2, \langle\cdot,\cdot\rangle\right)$ with $\langle\bar{\mathbf{x}}, \bar{\mathbf{y}}\rangle = x_1y_1 + x_2y_2$, $\bar{\mathbf{x}} = (x_1, x_2)$, $\bar{\mathbf{y}} = (y_1, y_2) \in \mathbb{R}^2$. Let $\bar{\mathbf{y}} = (1, 1)$ and $\Gamma, \gamma > 0$ with $\Gamma > \gamma$. Then, by (5.10), we deduce

$$\sqrt{2}\sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leq 2c\cdot\frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{5.11}$$

If $x_1 = \Gamma$, $x_2 = \gamma$, then

$$\langle\Gamma\bar{\mathbf{y}} - \bar{\mathbf{x}}, \bar{\mathbf{x}} - \gamma\bar{\mathbf{y}}\rangle = (\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0,$$

showing that the condition (5.7) is valid. Replacing x_1 and x_2 in (5.11), we deduce

$$\sqrt{2}\sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \leq 2c\frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{5.12}$$

If in (5.12) we choose $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon$ with $\varepsilon \in (0, 1)$, then we have

$$2\sqrt{1+\varepsilon^2} - 2 \leq 2c\frac{4\varepsilon^2}{2},$$

giving

$$\sqrt{1+\varepsilon^2} - 1 \leq 2c\varepsilon^2. \tag{5.13}$$

Finally, multiplying (5.13) with $\sqrt{1+\varepsilon^2}+1>0$ and thus dividing by ε^2, we deduce

$$1 \le 2c\left(\sqrt{1+\varepsilon^2}+1\right) \quad \text{for any} \quad \varepsilon \in (0,1). \tag{5.14}$$

Letting $\varepsilon \to 0+$ in (5.14) we get $c \ge \frac{1}{4}$, and the sharpness of the constant is proved. ∎

5.2. **Reverses of the Triangle Inequality.** The following reverse of the triangle inequality in inner product spaces holds [10].

Proposition 13. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x,a \in H$ and $r>0$. If $\|x-a\| \le r$, then we have the inequality*

$$0 \le \|x\|+\|a\|-\|x+a\| \le r. \tag{5.15}$$

Proof. Since

$$(\|x\|+\|a\|)^2-\|x+a\|^2 \le 2\left(\|x\|\,\|a\|-\operatorname{Re}\langle x,a\rangle\right),$$

then by Theorem 7 we deduce

$$(\|x\|+\|a\|)^2-\|x+a\|^2 \le r^2,$$

from where we obtain

$$\|x\|+\|a\| \le \sqrt{r^2+\|x+a\|^2} \le r+\|x+a\|,$$

giving the desired result (5.15). ∎

We may state the following result [10].

Proposition 14. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x,y \in H$, $M>m>0$ such that either*

$$\operatorname{Re}\langle My-x, x-my\rangle \ge 0,$$

or equivalently,

$$\left\|x-\frac{M+m}{2}y\right\| \le \frac{1}{2}(M-m)\|y\|,$$

holds. Then we have the inequality

$$0 \le \|x\|+\|y\|-\|x+y\| \le \frac{\sqrt{2}}{2}\cdot\frac{M-m}{\sqrt{M+m}}\|y\|. \tag{5.16}$$

Proof. By Theorem 8 for $\Gamma=M$, $\gamma=m$, we have the inequality

$$\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle \le \frac{1}{4}\cdot\frac{(M-m)^2}{(M+m)}\|y\|^2.$$

Then we may state that

$$\begin{aligned}(\|x\|+\|y\|)^2-\|x+y\|^2 &= 2\left(\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\right)\\ &\le \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\|y\|^2,\end{aligned}$$

from where we get

$$\|x\|+\|y\| \leq \sqrt{\frac{1}{2} \cdot \frac{(M-m)^2}{M+m}\|y\|^2+\|x+y\|^2}$$
$$\leq \|x+y\|+\frac{M-m}{\sqrt{2(M+m)}}\|y\|,$$

giving the desired inequality (5.16). ∎

5.3. **Integral Inequalities.** We provide now, by the use of certain results obtained above, some integral inequalities that may be used in practical applications.

Proposition 15. *Let $f,g \in L^2_\rho(\Omega,\mathbb{K})$ and $r>0$ with the property that*

$$|f(s)-g(s)| \leq r, \quad \text{for } \mu-a.e. \quad s \in \Omega.$$

Then we have the inequalities

$$0 \leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right|$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_\Omega \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s)\right|$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \int_\Omega \rho(s) \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s)$$
$$\leq \frac{1}{2}r^2. \tag{5.17}$$

The constant $\frac{1}{2}$ is best possible in (5.17).

The proof follows by Theorem 7, and we omit the details.

Proposition 16. *Let $f,g \in L^2_\rho(\Omega,\mathbb{K})$ and $\gamma,\Gamma \in \mathbb{K}$ so that $\Gamma \neq -\gamma$, and*

$$\operatorname{Re}\left[(\Gamma g(s)-f(s))\left(\overline{f(s)}-\overline{\gamma}\overline{g(s)}\right)\right] \geq 0, \quad \text{for} \quad \mu-a.e. \quad s \in \Omega.$$

Then we have the inequalities

$$0 \leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right| \tag{5.18}$$
$$\leq \left[\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s)\right]\right|$$

$$\leq \left[\int_{\Omega}\rho(s)|f(s)|^2 d\mu(s)\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\int_{\Omega}\rho(s) f(s)\overline{g(s)}d\mu(s)\right]$$
$$\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s).$$

The constant $\frac{1}{4}$ *is best possible.*

Remark 17. *If the space is real and we assume, for* $M > m > 0$, *that*

$$mg(s) \leq f(s) \leq Mg(s), \quad \text{for } \mu - a.e. \ s \in \Omega, \tag{5.19}$$

then, by (5.18), we deduce the inequality:

$$0 \leq \left[\int_{\Omega}\rho(s)|f(s)|^2 d\mu(s)\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s)\right]^{\frac{1}{2}} - \left|\int_{\Omega}\rho(s) f(s)\overline{g(s)}d\mu(s)\right|$$
$$\leq \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s).$$

The constant $\frac{1}{4}$ *is best possible.*

The following reverse of the triangle inequality for integrals holds.

Proposition 17. *Assume that the functions* $f, g \in L^2_{\rho}(\Omega, \mathbb{K})$ *satisfy (5.19). Then we have the inequality*

$$0 \leq \left(\int_{\Omega}\rho(s)|f(s)|^2 d\mu(s)\right)^{\frac{1}{2}} + \left(\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s)\right)^{\frac{1}{2}} - \left(\int_{\Omega}\rho(s)|f(s)+g(s)|^2 d\mu(s)\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{2}}{2}\cdot\frac{M-m}{\sqrt{M+m}}\left(\int_{\Omega}\rho(s)|g(s)|^2 d\mu(s)\right)^{\frac{1}{2}}.$$

The proof follows by Proposition 14.

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School of Computer Science & Mathematics, Victoria University, Melbourne, Australia

E-mail address: Sever.Dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html