Weighted Extended Mean Values

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Abstract

The author generalizes Stolarsky's Extended Mean Values to a four-parameter family of means

$$F(r,s;a,b;x,y) = E(r,s;ax,by)/E(r,s;a,b)$$

and investigates their monotonicity properties.

1 Introduction

The inequalities $\sqrt{xy} \le \frac{y-x}{\log y - \log x} \equiv L(x,y) \le \frac{x+y}{2}$, and the observation that for natural s the inequalities

$$\min(x,y) \le \left(\frac{x^s + x^{s-1}y + \cdots + xy^{s-1} + y^s}{s+1}\right)^{1/s} \le \max(x,y) \qquad (1)$$

hold, led Galvani [1] to the investigation of one-parameter family of means defined as

$$S_p(x,y) = \left(\frac{y^p - x^p}{p(y-x)}\right)^{1/(p-1)},\ S_0(x,y) = L(x,y),\ S_1(x,y) = e^{-1}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}.$$

Observe that for $p = -1$ and 2 we obtain the geometric and the arithmetic means. It has been proved that for $p < q$ $S_p(x,y) \le S_q(x,y)$ and that S_p is increasing in both variables. Stolarsky [5] and later Leach and Sholander [2, 3] extended this family to a two-parameter extended mean values

$$E(r,s;x,y) = \begin{cases} \left(\frac{r}{s}\frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)} & sr(s-r)(x-y) \ne 0, \\ \left(\frac{1}{r}\frac{y^r - x^r}{\log y - \log x}\right)^{1/r} & r(x-y) \ne 0,\ s = 0, \\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r - x^r)} & r = s,\ r(x-y) \ne 0, \\ \sqrt{xy} & r = s = 0,\ x - y \ne 0, \\ x & x = y. \end{cases} \qquad (2)$$

and proved that E is continuous and increasing in all variables. Interesting proof of this fact can be also found in [8].

In this paper we extend E to a four-parameter family of means and investigate their monotonicity properties.

The following inequalities

$$\min(x,y) \le \left(\frac{(ax)^s + (ax)^{s-1}by + \cdots + ax(by)^{s-1} + (by)^s}{a^s + a^{s-1}b + \cdots + ab^{s-1} + b^s} \right)^{1/s} \le \max(x,y) \tag{3}$$

valid for natural s and positive x, y, a, b, will be the departure point for our investigation.

We follow Stolarsky's way and define

$$F(r,s;a,b;x,y) = \left(\frac{(ax)^s - (by)^s}{a^s - b^s} \Big/ \frac{(ax)^r - (by)^r}{a^r - b^r} \right)^{1/(s-r)} \tag{4}$$

for $rs(r-s)(ax-by)(a-b) \neq 0$. Note that (4) can be written as

$$F(r,s;a,b;x,y) = \frac{E(r,s;ax,by)}{E(r,s;a,b)}, \tag{5}$$

thus extending F to a continuous function in $\mathbb{R}^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$.

In section 3 we show that F is a mean of x and y and is monotone in all variables though the monotonicity in r, s, a and b may not be the same for different values of other parameters.

2 Tools

Before formulating our main results we define some tools and prove a useful lemma.

For a function $f(x)$ we write $\mathrm{Mon}_x(f) = 1, 0, -1$ if f is increasing, constant or decreasing in x, respectively.

Similarly, $\mathrm{Con}_x\, f(x) = 1, 0, -1$ if f is convex, linear or concave in x.

We omit the subscript in case of functions of one variable. It is worth noting some basic properties of the operators Mon and Con:

$\mathrm{Mon}(f(g)) = \mathrm{Mon}(f)\,\mathrm{Mon}(g)$,
If $x = f(y)$ then $\mathrm{Mon}_x(g) = \mathrm{Mon}(f)\,\mathrm{Mon}_y(g(f))$,
$\mathrm{Con}(f) = \mathrm{Mon}(f') = \mathrm{sgn}(f'')$,
For fixed c and positive f $\mathrm{Mon}(f^c) = \mathrm{sgn}(c)\,\mathrm{Mon}(f)$,
$\mathrm{Con}_x(x^c) = \mathrm{sgn}(c(c-1))$,

$\mathrm{Mon}_x(x^c) = \mathrm{sgn}(c)$,
$\mathrm{sgn}(f(x) - f(y)) = \mathrm{Mon}(f)\,\mathrm{sgn}(x - y)$ for strictly monotone f.

Let us recall now two properties of convex function that will be extremely useful [4].

Property 1 *f is convex (concave) if an only if the 'difference quotient' function $\frac{f(x)-f(y)}{x-y}$, $x \neq y$, is increasing in both variables x and y.*

Property 2 *If f is convex and $z > 0$ ($z < 0$), then the function $g(x) = f(x+z) - f(x)$ is increasing (decreasing). For concave functions, the monotonicities reverse.*

The above properties can be written as

$$\mathrm{Con}(f) = \mathrm{Mon}_x\left(\frac{f(x)-f(y)}{x-y}\right) = \mathrm{Mon}_y\left(\frac{f(x)-f(y)}{x-y}\right), \tag{6}$$

$$\mathrm{Con}(f) = \mathrm{sgn}(z)\,\mathrm{Mon}_x(f(x+z) - f(x)). \tag{7}$$

Lemma 1 *If $A, B > 0,\ A, B \neq 1,\ A \neq B,\ A \neq B^{-1}$, then the function*

$$H(t) = \log\left|\frac{1-A^t}{1-B^t}\right|,\ H(0) = \log\left|\frac{\log A}{\log B}\right|$$

is strictly convex or concave and

$$\mathrm{Con}(H) = \mathrm{sgn}(\log^2 A - \log^2 B). \tag{8}$$

Proof.

$$\begin{aligned} H''(t) &= \frac{B^t \log^2 B}{(1-B^t)^2} - \frac{A^t \log^2 A}{(1-A^t)^2} \\ &= C(t)\left(\frac{A^t - 2 + A^{-t}}{\log^2 A} - \frac{B^t - 2 + B^{-t}}{\log^2 B}\right) \\ &= C(t)\sum_{k=2}^{\infty} \frac{(\log^2 A)^{k-1} - (\log^2 B)^{k-1}}{(2k)!} t^{2k} \\ &= C(t)(\log^2 A - \log^2 B)\sum_{k=2}^{\infty} \frac{\sum_{j=0}^{k-2} (\log^2 A)^j (\log^2 B)^{k-2-j}}{(2k)!} t^{2k}, \end{aligned} \tag{9}$$

where $C(t) = \frac{B^t \log^2 B}{(1-B^t)^2} \frac{A^t \log^2 A}{(1-A^t)^2}$ is positive. □

3 Monotonicity of $F(r, s; a, b; x, y)$.

Theorem 1 (Monotonicity in x and y)

$$\mathrm{Mon}_x(F) = \mathrm{Mon}_y(F) = 1.$$

Proof. The result follows immediately from (5) and monotonicity of E, but we will give here an independent proof.
Suppose first that $rs(r-s)(a-b)(x-y) \neq 0$ and write F as

$$\left(\frac{((ax)^r)^{s/r} - ((by)^r)^{s/r}}{(ax)^r - (by)^r} \frac{a^r - b^r}{a^s - b^s} \right)^{1/(s-r)}.$$

One can see immediately that F as a function of x is a composition of four monotone functions: $f_1(t) = (at)^r, f_2$ is the 'difference quotient' function obtained from $t^{s/r}$ (see Property 1), $f_3(t) = \frac{a^r - b^r}{a^s - b^s} t$, and $f_4(t) = t^{1/(s-r)}$.
So F is monotone and

$$\begin{aligned} \mathrm{Mon}_x F &= \mathrm{sgn}(r) \, \mathrm{Con}(t^{s/r}) \, \mathrm{sgn} \frac{a^r - b^r}{a^s - b^s} \, \mathrm{sgn} \frac{1}{s-r} \\ &= \mathrm{sgn} \left(r \frac{s}{r} \left(\frac{s}{r} - 1 \right) \frac{r}{s} \frac{1}{s-r} \right) = 1. \end{aligned}$$

If $r = 0$

$$F(s, 0) = F(0, s) = \left(\frac{\exp(s \log(ax)) - \exp(s \log(by))}{\log(ax) - \log(by)} \frac{\log a - \log b}{a^s - b^s} \right)^{1/s}$$

and we have similar situation with $f_1(t) = \log(at)$ and f_2 coming from e^{st}.
So

$$\mathrm{Mon}_x F = \mathrm{Mon}(f_1) \, \mathrm{Con}_t(e^{st}) \, \mathrm{sgn} \left(\frac{\log a - \log b}{a^s - b^s} \right) \mathrm{sgn} \frac{1}{s} = 1.$$

In the case $r = s$

$$\log F = -\frac{1}{s} + \frac{1}{s} \frac{(ax)^s \log(ax)^s - (by)^s \log(by)^s}{(ax)^s - (by)^s} - \log E(s, s, a, b)$$

is monotone in x for the same reason as above, and

$$\mathrm{Mon}_x(F) = \mathrm{Mon}_x(\log F) = \mathrm{Mon}_t(t^s) \, \mathrm{Con}(s^{-1} t \log t) = 1.$$

We leave the case $a = b$ to the reader.
Proof of the monotonicity in y is exactly the same. □

Theorem 2 (Monotonicity in r and s)

$$\mathrm{Mon}_r(F) = \mathrm{Mon}_s(F) = \mathrm{sgn}(x-y)\,\mathrm{sgn}(a^2x-b^2y). \tag{10}$$

Proof. We consider four cases:
Case 1: $x = y$ or $a^2x = b^2y$.
In this case the right hand side of (10) equals 0. An easy calculation shows that

$$F(r,s;a,b;x,y) = \begin{cases} x & \text{if } x = y, \\ \sqrt{ab^{-1}xy} & \text{if } a^2x = b^2y \end{cases} \tag{11}$$

is constant in r and s, so our theorem holds.

Case 2: $a = b$.
The right hand side of (10) equals 1 and from (2) and (5)

$$\begin{aligned} \log F(r,s;a,a;x,y) &= \log E(r,s;x,y) \\ &= \frac{\log\left|\frac{y^s-x^s}{s}\right| - \log\left|\frac{y^r-x^r}{r}\right|}{s-r}. \end{aligned} \tag{12}$$

As the function $f(s) = \log\left|\frac{y^s-x^s}{s}\right|$ is convex (proof is almost the same as the proof of Lemma 1), it follows from (12) and property 2 that $\log F$ and $\log E$, hence F as well as E are increasing in r and s.

Case 3: $ax = by$.
$\mathrm{sgn}((x-y)(a^2x-b^2y)) = \mathrm{sgn}((x-y)(ax)^2(x^{-1}-y^{-1})) = -1$. By (12) and (5)

$$F(r,s;a,b;x,y) = \frac{\sqrt{abxy}}{E(r,s;a,b)}$$

hence from $\mathrm{Mon}_{r,s}(E) = 1$ it follows that $\mathrm{Mon}_{r,s}(F) = -1$.

Case 4: all other cases.

$\operatorname{sgn}(x-y)\operatorname{sgn}(a^2x-b^2y) =$

$$
\begin{aligned}
&= \operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}\left(\log\frac{a^2x}{b^2y}\right)\\
&= \operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}\left(2\log\frac{a}{b}+\log\frac{x}{y}\right)\\
&= \operatorname{sgn}\left(\log^2\frac{ax}{by}-\log^2\frac{a}{b}\right)\\
&= \operatorname{Con}_t\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^t}{1-\left(\frac{a}{b}\right)^t}\right|\right) && \text{(by Lemma 1)}\\
&= \operatorname{Mon}_{r,s}\left(\frac{1}{s-r}\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^s}{1-\left(\frac{a}{b}\right)^s}\right|-\log\left|\frac{1-\left(\frac{ax}{by}\right)^r}{1-\left(\frac{a}{b}\right)^r}\right|\right)\right) && \text{(by (6))}\\
&= \operatorname{Mon}_{r,s}(-\log y+\log F) = \operatorname{Mon}_{r,s}(F).
\end{aligned}
$$

□

Theorem 3 (Monotonicity in a and b)

$$\operatorname{Mon}_a(F) = -\operatorname{Mon}_b(F) = \operatorname{sgn}(x-y)\operatorname{sgn}(r+s).$$

Proof. First observe that $F(r,-r;a,b;x,y)=\sqrt{xy}$, so the theorem holds if the right hand side equals 0.
For $r \neq s$
$\operatorname{sgn}(x-y)\operatorname{sgn}(r+s) =$

$$
\begin{aligned}
&= \operatorname{sgn}(x-y)\operatorname{sgn}(s-r)\operatorname{sgn}(s^2-r^2)\\
&= \operatorname{sgn}(s-r)\operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}(\log^2 e^s-\log^2 e^r)\\
&= \operatorname{sgn}(s-r)\operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{Con}_t\left(\log\left|\frac{1-e^{st}}{1-e^{rt}}\right|\right) && (13)\\
&= \operatorname{sgn}(s-r)\operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}(z)\operatorname{Mon}_t\left(\log\left|\frac{1-e^{s(t+z)}}{1-e^{r(t+z)}}\right|-\log\left|\frac{1-e^{st}}{1-e^{rt}}\right|\right), && (14)
\end{aligned}
$$

where (13) and (14) follow from Lemma 1 and Property 2.
Let $z=\log(x/y)$ and $t=\log(a/b)$. Note that $\operatorname{Mon}_t(a) = -\operatorname{Mon}_t(b) = 1$, and (14) transforms into

$\operatorname{sgn}(x-y)\operatorname{sgn}(r+s) =$

$$= \operatorname{sgn}(s-r)\operatorname{Mon}_t(a)\operatorname{Mon}_a\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^s}{1-\left(\frac{ax}{by}\right)^r}\right| - \log\left|\frac{1-\left(\frac{a}{b}\right)^s}{1-\left(\frac{a}{b}\right)^r}\right|\right)$$
$$= \operatorname{sgn}(s-r)\operatorname{Mon}_a(\log y^{r-s} + \log F^{s-r}) = \operatorname{Mon}_a(F)$$

or

$$= \operatorname{sgn}(s-r)\operatorname{Mon}_t(b)\operatorname{Mon}_b\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^s}{1-\left(\frac{ax}{by}\right)^r}\right| - \log\left|\frac{1-\left(\frac{a}{b}\right)^s}{1-\left(\frac{a}{b}\right)^r}\right|\right)$$
$$= -\operatorname{sgn}(s-r)\operatorname{Mon}_b(\log y^{r-s} + \log F^{s-r}) = -\operatorname{Mon}_b(F).$$

The case $s = r$ can be shown using continuity of F. □

Theorem 4

$$\min(x, y) \le F(r, s; a, b; x, y) \le \max(x, y).$$

Proof. As F is monotone in a it is enough to show that $\lim_{a\to 0} F$ and $\lim_{a\to\infty} F$ satisfy the same inequality. But

$$\lim_{a\to 0} F = \sqrt{xy}\left(\sqrt{\frac{y}{x}}\right)^{\frac{r+s}{|r|+|s|}} \text{ and } \lim_{a\to\infty} F = \sqrt{xy}\left(\sqrt{\frac{x}{y}}\right)^{\frac{r+s}{|r|+|s|}}$$

which completes the proof. □

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