A NEW ALGEBRAIC INEQUALITY AND ITS APPLICATION

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Abstract. In this short note, we give a new algebraic inequality, and by its application, we generalize a known result in [1].

1. Introduction and Main Result

In 2003, the following algebraic inequality is proved by A.-L. Liu [1]:

Theorem 1.1. *Let* $a_i, 1 \leqslant i \leqslant 10$ *be non-negative real numbers for* $\sum_{i=1}^{10} a_i = 30$*, then*

$$\sum_{i=1}^{10}(a_i - 1)(a_i - 2)(a_3 - 3) \geqslant 0 \tag{1.1}$$

In this short note, we give a new theorem for algebraic inequality, and by its generalization, we generalize inequality (1.1).

Theorem 1.2. *Let* p, q *be integers for* $p + q > 0$*, then the inequality*

$$\prod_{j=-q}^{p}(a - j) \geqslant (p + q)!(a - p) \tag{1.2}$$

holds if $p+q$ *be a even number and* a *be a real number or* $p+q$ *be a odd number and* $a+q+1 > 0$*; and the reversed inequality holds if* $p + q$ *be a odd number and* $a + q + 1 < 0$*. With equality holding if and only if* $a = p$*.*

From Theorem1.2, we easily prove the following generalization of Theorem1.1:

Theorem 1.3. *Let* p, q *be integers for* $p + q > 0$*, and* $m, a_i, 1 \leqslant i \leqslant n$ *be real numbers for* $\sum_{i=1}^{n} a_i = m \cdot n$*. If* $p \leqslant [m]$*,* $p + q$ *be a even number and* a *be a real number or* $p + q$ *be a odd number and* $a_i + q + 1 > 0 (1 \leqslant i \leqslant n)$*, then*

$$\sum_{i=1}^{n}\prod_{j=-q}^{p}(a_i - j) \geqslant 0 \tag{1.3}$$

and the reversed inequality holds if $p + q$ *be a odd number and* $a_i + q + 1 < 0 (1 \leqslant i \leqslant n)$*. With equality holding if and only if* $p = m = a_i, 1 \leqslant i \leqslant n$*.*

2. Lemma

Lemma 2.1. *Let*

$$u(x) = \prod_{j=1}^{k}(x - j) - k! \tag{2.1}$$

then the equation $u(x) = 0$ *which have only real roots* $x_1 = 0, x_2 = k + 1$ *if* k *be a even number, and which have only real root* $x_1 = k + 1$ *if* k *be a odd number.*

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Proof. Denote $\varphi(x) = \prod_{j=1}^{k}(x-j)$, then the equations (2.1) and

$$u(x) = \varphi(x) - k! = 0 \tag{2.2}$$

are equivalence.

To solve the equation (2.2), we firstly give the real roots of equation

$$\mid \varphi(x) \mid - k! = 0. \tag{2.3}$$

The equation (2.3) which have and only have real roots $x_1 = 0, x_2 = k+1$, since the following (i)-(iii):

(i) By all appearances, real numbers $x_1 = 0, x_2 = k+1$ are the roots of equation (2.3).

(ii) When $x > k+1$, we have $\mid x - i \mid > k + 1 - i$ $(i = 1, 2, \cdots, k)$, and when $x < 0$, we obtain $\mid x - i \mid > i$ $(i = 1, 2, \cdots, k)$, these both find $\mid \varphi(x) \mid > k!$.

(iii) When $0 < x < k+1$, we have i in addition $i - 1 < x < i + 1$ $(i = 1, 2, \cdots, k)$, and $\mid x-1 \mid < i, \mid x-2 \mid < i-1, \cdots, \mid x-i \mid < 1, \mid x-(i+1) \mid < i+1, \mid x-k \mid < k$, these are $\mid \varphi(x) \mid < k!$.

The Lemma2.1 is proved, because $\mid \varphi(x) \mid - k! = 0 \Longleftrightarrow \varphi(x) - k! = 0$ or $\varphi(x) + k! = 0$. ∎

From Lemma2.1 in $k = p+q$, where p, q be integers for $p+q > 0$, we have

Remark 2.1. *Let*

$$u(x^{,}) = \prod_{j=1}^{p+q}(x^{,} - j) - (p+q)! = \prod_{j=-q}^{p-1}(x^{,} - q - 1 - j) - (p+q)! \tag{2.4}$$

or

$$v(x) = \prod_{j=-q}^{p-1}(x-j) - (p+q)! \tag{2.5}$$

then the equation $v(x) = 0$ which have only real roots $x_1 = -q-1, x_2 = p$ if $p+q$ be a even number, and which have only real root $x_1 = p$ if $p+q$ be a odd number.

3. The Proof of Theorem1.2

Now, we give the following proof of Theorem1.2:

Proof. Set

$$f(x) = \prod_{j=-q}^{p}(x-j) - (p+q)!(x-p) = (x-p)v(x) \tag{3.1}$$

where $v(x) = \prod_{j=-q}^{p-1}(x-j) - (p+q)!$.

From Remark2.4, we obtain

$$F(x) = \begin{cases} (x-p)^2 g(x) & if\ p+q\ be\ a\ even\ number, \\ (x+q+1)(x-p)^2 h(x) & if\ p+q\ be\ a\ odd\ number. \end{cases} \tag{3.2}$$

We easily know $g(x) > 0$ and $h(x) > 0$, because $g(x) = 0$ and $h(x) = 0$ both have'nt real root.

Therefore

$$F(a) \geqslant 0 \begin{cases} if\ p+q\ be\ a\ even\ number\ and\ a\ be\ a\ real\ number, \\ if\ p+q\ be\ a\ odd\ number,\ and\ a+q+1 > 0. \end{cases} \tag{3.3}$$

The proof of Theorem1.3 is completed.

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References

[1] A.-L. Liu. *The Solution of Mathematical Problem 1412.* Shuxueongbao, Beijin, (1)2003.

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