SOME COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA AND POLYGAMMA FUNCTIONS

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Abstract. The function $\frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ is strictly logarithmically completely monotonic in $(0,\infty)$. The function $\psi''(x+2)+\frac{1+x^2}{x^2(1+x)^2}$ is strictly completely monotonic in $(0,\infty)$.

1. Introduction

It is well known that the gamma function $\Gamma(z)$ is defined for $\operatorname{Re} z>0$ as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \tag{1}$$

The psi or digamma function $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed for $x>0$ and $k\in\mathbb{N}$ as

$$\psi(x)=-\gamma+\sum_{n=0}^\infty\left(\frac{1}{1+n}-\frac{1}{x+n}\right), \tag{2}$$

$$\psi^{(k)}(x)=(-1)^{k+1}k!\sum_{i=0}^\infty\frac{1}{(x+i)^{k+1}}, \tag{3}$$

$$\psi(x)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{4}$$

$$\psi^{(k)}(x)=(-1)^{k+1}\int_0^\infty\frac{t^ke^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{5}$$

where $\gamma=0.57721566490153286\cdots$ is the Euler-Mascheroni constant.

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A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0 \tag{6}$$

for $x \in I$ and $n \geq 0$. If inequality (6) is strict for all $x \in I$ and for all $n \geq 0$, then f is said to be strictly completely monotonic.

For $x > 0$ and $s \geq 0$, we have

$$\frac{1}{(x+s)^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(x+s)t} \, \mathrm{d}t, \quad n \in \mathbb{N}. \tag{7}$$

A function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \tag{8}$$

for $k \in \mathbb{N}$ on I. If inequality (8) is strict for all $x \in I$ and for all $k \in \mathbb{N}$, then f is said to be strictly logarithmically completely monotonic.

In [4] it is proved that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic. But not conversely, since a convex function may not be logarithmically convex (see Remark. 1.16 at page 7 in [3]).

Completely monotonic functions have applications in many branches. For example, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics. Some related references are listed in [1].

It is well known that the function $\left(1 + \frac{1}{x}\right)^{-x}$ is strictly completely monotonic in $(0, \infty)$. In [1], it is proved that the function $\left(1 + \frac{a}{x}\right)^{x+b} - e^a$ is completely monotonic with $x \in (0, \infty)$ if and only if $a \leq 2b$, where $a > 0$ and b are real numbers.

Among other things, the following completely monotonic properties are obtained in [4]: For $\alpha \leq 0$, the function $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$ is strictly completely monotonic in $(0, \infty)$. For $\alpha \geq 1$, the function $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ is strictly completely monotonic in $(0, \infty)$.

In [2] the following two inequalities are presented: For $x \in (0, 1)$, we have

$$\frac{x}{[\Gamma(x+1)]^{1/x}} < \left(1 + \frac{1}{x}\right)^x < \frac{x+1}{[\Gamma(x+1)]^{1/x}}. \tag{9}$$

For $x \geq 1$,

$$\left(1+\frac{1}{x}\right)^x \geq \frac{x+1}{[\Gamma(x+1)]^{1/x}}. \tag{10}$$

Equality in (10) occurs for $x = 1$.

It is easy to see that

$$\lim_{x\to\infty} \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x = 1. \tag{11}$$

The main purpose of this paper is to give a strictly logarithmically completely monotonic property of the function $\frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ in $(0,\infty)$ as follows.

Theorem 1. *The function* $\frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x$ *is strictly logarithmically completely monotonic in* $(0,\infty)$.

As a direct consequence of the proof of Theorem 1, we have the following

Corollary 1. *The function*

$$\psi''(x) + \frac{x^4+5x^3+7x^2+7x+2}{x^3(x+1)^3} = \psi''(x+2) + \frac{1+x^2}{x^2(1+x)^2} \tag{12}$$

is strictly completely monotonic in $(0,\infty)$.

2. Proof of Theorem 1

Define

$$F(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c}\left(1+\frac{a}{x}\right)^{x+b} \tag{13}$$

for $x > 0$ and some fixed real numbers a, b and c.

Taking the logarithm of $F(x)$ defined by (13) and differentiating yields

$$\ln F(x) = (x+b)\ln\left(1+\frac{a}{x}\right) + \frac{\ln\Gamma(x+1)}{x} - c\ln x, \tag{14}$$

$$[\ln F(x)]' = \ln\left(1+\frac{a}{x}\right) - \frac{a(x+b)}{x(x+a)} + \frac{x\psi(x+1)-\ln\Gamma(x+1)}{x^2} - \frac{c}{x}, \tag{15}$$

and

$$\begin{aligned}[\ln F(x)]^{(n)} &= (-1)^{n-1}(n-1)!(x+b)\left[\frac{1}{(x+a)^n} - \frac{1}{x^n}\right] \\ &\quad + (-1)^n(n-2)!n\left[\frac{1}{(x+a)^{n-1}} - \frac{1}{x^{n-1}}\right] \\ &\quad + \frac{h_n(x)}{x^{n+1}} + (-1)^n(n-1)!\frac{c}{x^n}\end{aligned}$$

$$= (-1)^n (n-2)!\left[\frac{(n-1)(b+c)-x}{x^n} + \frac{x+na-(n-1)b}{(x+a)^n}\right] + \frac{h_n(x)}{x^{n+1}}, \quad (16)$$

where $n \ge 2$, $\psi^{(-1)}(x+1) = \ln\Gamma(x+1)$, $\psi^{(0)}(x+1) = \psi(x+1)$, and

$$h_n(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!}, \quad (17)$$

$$h_n'(x) = x^n \psi^{(n)}(x+1) \begin{cases} > 0, & \text{if } n \text{ is odd}, \\ < 0, & \text{if } n \text{ is even}. \end{cases} \quad (18)$$

Therefore, we have

$$\begin{aligned} &(-1)^n x^{n+1} [\ln F(x)]^{(n)} \\ &= (n-2)!\left\{(n-1)(b+c) - x + \frac{x^n[x+na-(n-1)b]}{(x+a)^n}\right\}x + (-1)^n h_n(x) \quad (19) \end{aligned}$$

and

$$\begin{aligned} &\frac{\mathrm{d}\big\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\big\}}{\mathrm{d}x} \\ &= (-1)^n x^n \psi^{(n)}(x+1) + (n-2)!\bigg\{(n-1)(b+c) - 2x \\ &\quad + \frac{x^n[a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2]}{(x+a)^{n+1}}\bigg\} \\ &= x^n\bigg\{(-1)^n\psi^{(n)}(x+1) + (n-2)!\bigg[\frac{(n-1)(b+c)-2x}{x^n} \\ &\quad + \frac{a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2}{(x+a)^{n+1}}\bigg]\bigg\} \\ &= x^n\bigg\{(-1)^n\psi^{(n)}(x) + \frac{n!}{x^{n+1}} + (n-2)!\bigg[\frac{(n-1)(b+c)-2x}{x^n} \\ &\quad + \frac{a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2}{(x+a)^{n+1}}\bigg]\bigg\}. \end{aligned}$$

By letting $a = c = 1$ and $b = 0$, we have

$$\begin{aligned} \frac{\mathrm{d}\big\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\big\}}{\mathrm{d}x} &= x^n\bigg\{(-1)^n\psi^{(n)}(x) + \frac{n!}{x^{n+1}} \\ &\quad + (n-2)!\bigg[\frac{n-1-2x}{x^n} + \frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\} \\ &= x^n\bigg\{(-1)^n\psi^{(n)}(x) + (n-2)!\bigg[\frac{n(n-1)+(n-1)x-2x^2}{x^{n+1}} \\ &\quad + \frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\} \end{aligned}$$

$$\triangleq x^n\big\{(-1)^n\psi^{(n)}(x)+(n-2)!g_n(x)+(n-2)!h_n(x)\big\}.$$

By induction, it follows that

$$g_n'(x)=-(n-1)g_{n+1}(x) \quad \text{and} \quad h_n'(x)=-(n-1)h_{n+1}(x), \tag{20}$$

this implies

$$g_2^{(n-2)}(x)=(-1)^n(n-2)!g_n(x) \quad \text{and} \quad h_2^{(n-2)}(x)=(-1)^n(n-2)!h_n(x), \tag{21}$$

therefore

$$\frac{\mathrm{d}\big\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\big\}}{\mathrm{d}x}=(-1)^n x^n\big[\psi''(x)+g_2(x)+h_2(x)\big]^{(n-2)}. \tag{22}$$

From formulas (3), (5) and (7), for $x\in(0,\infty)$ and any nonnegative integer i, we have

$$\begin{aligned}
\phi(x)&\triangleq \psi''(x)+g_2(x)+h_2(x)\\
&=\psi''(x)+\frac{2+x-2x^2}{x^3}+\frac{2(3+3x+x^2)}{(x+1)^3}\\
&=\psi''(x)+\frac{x^4+5x^3+7x^2+7x+2}{x^3(x+1)^3}\\
&=\psi''(x)+\frac{2}{x^3}+\frac{1}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^3}+\frac{2}{(1+x)^2}+\frac{2}{1+x}\\
&=\frac{1}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^2}+\frac{2}{1+x}-2\sum_{i=2}^{\infty}\frac{1}{(x+i)^3}\\
&=\psi''(x+2)+\frac{1}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^2}+\frac{2}{1+x}\\
&=\psi''(x+2)+\frac{1+x^2}{x^2(1+x)^2}\\
&=\int_0^\infty te^{-xt}\,\mathrm{d}t-2\int_0^\infty e^{-xt}\,\mathrm{d}t+2\int_0^\infty te^{-(x+1)t}\,\mathrm{d}t\\
&\quad+2\int_0^\infty e^{-(x+1)t}\,\mathrm{d}t-\int_0^\infty\frac{t^2e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t\\
&=\int_0^\infty\big[t-2+(t+4)e^{-t}-(t^2+2t+2)e^{-2t}\big]e^{-xt}\,\mathrm{d}t\\
&\triangleq\int_0^\infty q(t)e^{-xt}\,\mathrm{d}t,\\
\phi^{(i)}(x)&=(-1)^i\int_0^\infty q(t)t^ie^{-xt}\,\mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned} q'(t) &= \big(2+2t+2t^2-3e^t+e^{2t}-te^t\big)e^{-2t} \\ &\triangleq p(t)e^{-2t}, \\ p'(t) &= 2+4t-4e^t+2e^{2t}-te^t, \\ p''(t) &= 4-5e^t+4e^{2t}-te^t, \\ p'''(t) &= \big(8e^t-t-6\big)e^t \\ &> 0. \end{aligned}$$

Hence, $p''(t)$ increases in $(0,\infty)$. Since $p''(0)=3>0$, we have $p''(t)>0$ and $p'(t)$ is increasing. Because of $p'(0)=0$, it follows that $p'(t)>0$ in $(0,\infty)$, and then $p(t)$ is increasing. From $p(0)=0$, it is deduced that $p(t)>0$ and $q'(t)>0$ in $(0,\infty)$, then $q(t)$ increases. As a result of $q(0)=0$, we obtain $q(t)>0$ in $(0,\infty)$. Therefore, we have $\phi(x)>0$ in $(0,\infty)$, and then for all nonnegative integer i, we have $(-1)^i\phi^{(i)}(x)>0$ in $(0,\infty)$. This means that the function $\psi''(x)+g_2(x)+h_2(x)$ is strictly completely monotonic on $(0,\infty)$.

Thus the function $(-1)^n x^{n+1}[\ln F(x)]^{(n)}$ is increasing in $x\in(0,\infty)$. Since

$$\lim_{x\to 0}\big\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\big\}=0,$$

we have $(-1)^n x^{n+1}[\ln F(x)]^{(n)}>0$, then $(-1)^n[\ln F(x)]^{(n)}>0$ for $n\ge 2$ in $(0,\infty)$. Since $[\ln F(x)]''>0$, the function $[\ln F(x)]'$ is increasing. It is not difficult to obtain $\lim_{x\to\infty}[\ln F(x)]'=0$, so $[\ln F(x)]'<0$ and $\ln F(x)$ is decreasing in $(0,\infty)$. In conclusion, the function $\ln F(x)$ is strictly completely monotonic in $(0,\infty)$. The proof is complete.

3. An open problem

Open Problem. *Under what conditions on a, b and c the function $F(x)$ defined by* (13) *is strictly logarithmically completely monotonic in $(0,\infty)$?*

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