ON A FURTHER IMPROVEMENT OF THE EXTENDED HARDY-HILBERT'S INEQUALITY

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Abstract. In this paper, by means of a sharpening of Hölder's inequality, the extended Hardy-Hilbert's inequalities with parameters A, B, λ are improved.

1. Introduction

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n > 0, b_n > 0$, if $0 < \sum_{n=1}^{\infty} a_n^p < +\infty$, $0 < \sum_{n=1}^{\infty} b_n^q < +\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{sin(\pi/p)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q} \tag{1.1}$$

the inequality (1.1) is known as the famous Hardy-Hilbert's inequality, it is important in analysis and its applications. The corresponding integral form of (1.1) can be stated as follows:

$$\text{Suppose that } p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad f, g > 0, \text{ if } \quad 0 < \int_0^{\infty} f^p(t)dt < +\infty,$$

$$0 < \int_0^{\infty} g^q(t)dt < +\infty \qquad \text{then}$$

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(x)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(t)dt\right)^{\frac{1}{q}} \tag{1.2}$$

Where the constant $\frac{\pi}{\sin(\pi/p)}$ are best possible in (1.1) and (1.2). In recent years, the inequalities (1.1) and (1.2) were studied extensively, and some improvements and extensions of Hilbert's inequality and Hardy-Hilbert's inequality with numerous variants have been given in many literature, see [2-4,6-8].
For example, Gao [2]proved the following Hilbert inequality:

$$\left[\int_0^{\infty}\int_0^{\infty}\frac{f(s)g(t)}{s+t}dsdt\right]^2 < \pi^2\int_0^{\infty} f^2(t)dt\int_0^{\infty} g^2(t)dt - G(\xi, \eta, \delta) \tag{1.3}$$

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Yang and Debnath [3] gave a generalization of Hardy-Hilbert's inequality as follows: For $\lambda > 2 - min\{p,q\}$ then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda} dxdy < \frac{K_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\int_0^\infty x^{(1-\lambda)} f^p(x) dx \right)^{1/p} \left(\int_0^\infty x^{(1-\lambda)} g^q(x) dx \right)^{1/q} \tag{1.4}$$

For $2 - min\{p,q\} < \lambda \le 2$, then

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(Am+Bn)^\lambda} < \frac{K_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left\{ \sum_{n=1}^\infty n^{(1-\lambda)} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^\infty n^{(1-\lambda)} b_n^q \right\}^{1/q} \tag{1.5}$$

where $\varphi_\lambda(r) = \frac{r+\lambda-2}{r}$ $(r = p,q), k_\lambda(p) = B(\varphi_\lambda(p), \varphi_\lambda(q))$. The main purpose of this paper is to establish a few new inequalities, and which are the extension of (1.3) and the improvements of the inequalities (1.4),(1.5). Furthermore, the results of the paper include the generalizations and improvements of corresponding ones in [2-3,6].

2. Lemmas and Their proofs

For convenience, we firstly introduce some notations:

$$(a^r, b^s) = \sum_{n=1}^\infty a_n^r b_n^s, \quad ||a||_p = \left(\sum_{n=1}^\infty a_n^p \right)^{1/p}, \quad ||a||_2 = ||a||$$

$$(f^r, g^s) = \int_0^{+\infty} f^r(x) g^s(x) dx \qquad ||f||_p = \left(\int_0^{+\infty} f^p(x) dx \right)^{\frac{1}{p}} \qquad ||f||_2 = ||f||$$

We next introduce a function defined by

$$S_r(H,x) = \left(H^{r/2}, x \right) ||H||_r^{-r/2}$$

where x is a parametric variable vector which is a variable unit vector, Under general case, it is properly chosen such that the specific problems discussed are simplified.

Clearly, $S_r(H,x) = 0$ when the vector x selected is orthogonal to $H^{p/2}$. Throughout this paper, the exponent k indicates $k = min\left\{ \frac{1}{p}, \frac{1}{q} \right\}$

In order to verify our assertions, we need to build the following lemmas.

Lemma 2.1. Let $f(x), g(x) > 0$. $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < ||f||_p < +\infty$, $0 < ||g||_q < +\infty$, then

$$(f,g) < ||f||_p ||g||_q (1-R)^k \tag{2.1}$$

where $R = (S_p(f,h) - S_q(g,h))^2, ||h|| = 1$, $f^{p/2}(x), g^{q/2}(x)$ and $h(x)$ are linearly independent.

Lemma 2.2 Let $a_n, b_n \ge 0$. $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|a\|_p < +\infty, 0 < \|b\|_q < +\infty$, then

$$(a,b) < ||a||_p ||b||_q (1-R)^k \tag{2.2}$$

where $R = (S_p(a,c) - S_q(b,c))^2 \neq 0, ||c|| = 1, a^{p/2}, b^{q/2}$ and c are linearly independent.
Lemma 2.1 and Lemma 2.2 are proved by [4], it is omitted here.
The choice of h and c are quite flexible, so long as condition $\|h\| = 1$ and $c = 1$ is satisfied, and on which we can refer to [3,7] etc.
In the following,we define

$$\varphi_\lambda(r) = \frac{r+\lambda-2}{r} \quad (r = p, q), \text{and} \quad k_\lambda(p) = B(\varphi_\lambda(p), \varphi_\lambda(q)), \tag{2.3}$$

$$\text{where} \quad B(u,v) = \int_0^\infty \frac{t^{-1+u}}{(1+t)^{u+v}} dt \quad (u,v>0) \quad \text{is beta function.}$$

$$D_r(A,B) = k_\lambda(p) A^{1-\lambda-\frac{2-\lambda}{r}} B^{\frac{2-\lambda}{r}-1} \tag{2.4}$$

Lemma 2.3 Let $\frac{1}{p} + \frac{1}{q} = 1, p > 1, \lambda > 2 - min(p,q), (r = p, q), m \in N$. Define the weight function $\omega_\lambda(A,B,r,x)$ and $\overline{\omega_\lambda}(A,B,r,m)$ as follows:

$$\omega_\lambda(A,B,r,x) = \int_0^\infty \frac{1}{(Ax+By)^\lambda} \left(\frac{x}{y}\right)^{\frac{2-\lambda}{r}} dy \tag{2.5}$$

$$\overline{\omega}_\lambda(A,B,r,m) = \sum_{n=1}^\infty \frac{1}{(Am+Bn)^\lambda} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{r}} \tag{2.6}$$

then we have

$$\omega_\lambda(A,B,r,x) = k_\lambda(p) x^{1-\lambda} A^{1-\lambda-\frac{2-\lambda}{r}} B^{\frac{2-\lambda}{r}-1} \tag{2.7}$$

For $0 \leq 2 - min\{p,q\} < \lambda \leq 2$,then

$$\overline{\omega}_\lambda(A,B,r,m) < \omega_\lambda(A,B,r,m) \tag{2.8}$$

The proof of the lemma is given by [3].

Lemma 2.4 Under the same condition as Lemma 3. Define the function $\theta_r(A,B,m)$ by

$$\theta_r(A,B,m) = \int_0^1 H_r(A,B,m,y)dy - \frac{m^{(2-\lambda)/r}}{2(Am+B)^\lambda} - \int_1^\infty \rho(y) H_r'(A,B,m,y)dy \tag{2.9}$$

where $\rho(y) = y - [y] - \frac{1}{2}$, and the function H_r is defined by

$$H_r(A,B,m,y) = \frac{1}{(Am+By)^\lambda} \left(\frac{m}{y}\right)^{(2-\lambda)/r} \tag{2.10}$$

then we have

$$\overline{\omega}_\lambda(A,B,r,m) = D_r(A,B) m^{1-\lambda} - \theta_r(A,B,m) \tag{2.11}$$

Proof: Consider the function H_r defined by (2.10). According to the paper [5], we have

$$\sum_{k=n+1}^{m} H_r(k) = \int_n^m H_r(y)dy + \frac{1}{2}(H_r(m) - H_r(n)) + \int_n^m \rho(y)H_r'(y)dy$$

Let $m \to \infty$ and $n = 1$. Then we obtain the Euler-Maclaurin summation formula of the form:

$$\sum_{k=1}^{\infty} H_r(k) = \int_1^\infty H_r(y)dy + \frac{1}{2}H_r(1) + \int_1^\infty \rho(y)H_r'(y)dy \tag{2.12}$$

We may apply the formula (2.12) to compute the weight function $\overline{\omega_\lambda}$ in (2.6)

$$\begin{aligned} &\overline{\omega}_\lambda(A,B,r,m) \\ &= \int_0^\infty H_r(y)dy - \left(\int_0^1 H_r(y)dy - \frac{1}{2}H_r(1) - \int_1^\infty \rho(y)H_r'(y)dy\right) \\ &= \omega_\lambda(A,B,r,m) - \left(\int_0^1 H_r(y)dy - \frac{m^{(2-\lambda)/r}}{2(Am+B)^\lambda} - \int_1^\infty \rho(y)H_r'(y)dy\right) \end{aligned} \tag{2.13}$$

Substituting (2.7),(2.4) and (2.9) into (2.13), the inequality (2.11) follows. By computation we can get $\theta_r(A,B,m) > 0$. Thus the lemma is proved.

3. Main Results

For the sake of convenient statement, we need again to define the functions and to introduce some notations

$$F = \frac{f(x)}{(Ax+By)^{\lambda/p}}\left(\frac{x}{y}\right)^{(2-\lambda)/pq}, \qquad G = \frac{g(y)}{(Ax+By)^{\lambda/q}}\left(\frac{y}{x}\right)^{(2-\lambda)/pq}$$

$$\alpha = \frac{a_m}{(Am+Bn)^{\lambda/p}}\left(\frac{m}{n}\right)^{(2-\lambda)/pq}, \qquad \beta = \frac{b_n}{(Am+Bn)^{\lambda/q}}\left(\frac{m}{n}\right)^{(2-\lambda)/pq},$$

$$(\alpha^r, \gamma^s) = \sum_{m=1}^\infty \sum_{n=1}^\infty \alpha^r \gamma^s, \qquad (F^r, h^s) = \int_0^\infty \int_0^\infty F^r\, h^s dxdy$$

$$E = \int_0^\infty x^{1-\lambda}e^{-2x}dx \tag{3.1}$$

$$\Psi(y) = \left\{\frac{1}{D_q(A,B)E}\right\}^{1/2} \int_0^\infty \frac{e^{-x}}{(Ax+By)^\lambda}\left(\frac{y}{x}\right)^{(2-\lambda)(q-p)/(2pq)} dx \tag{3.2}$$

$$S_p(F,h) = E^{-1/2}\left\{\int_0^\infty x^{1-\lambda}e^{-x}f^{p/2}(x)dx\right\} \cdot \left\{\int_0^\infty x^{1-\lambda}f^p(x)dx\right\}^{-1/2}$$

$$S_q(G,h) = \left\{\int_0^\infty \Psi(y)g^{q/2}(y)dy\right\} \cdot \left\{\int_0^\infty D_p(B,A)x^{1-\lambda}g^q(x)dx\right\}^{-1/2}$$

$$S_p(\alpha,\gamma) = \left\{\frac{a_1^p}{(A+B)^\lambda}\right\}\left\{\sum_{m=1}^\infty \overline{\omega}_\lambda(A,B,q,m)a_m^p\right\}^{-1/2}$$

$$S_q(\beta,\gamma) = \left\{\frac{b_1^q}{(A+B)^\lambda}\right\}\left\{\sum_{m=1}^\infty \overline{\omega}_\lambda(B,A,p,m)b_m^q\right\}^{-1/2}$$

where $\overline{\omega}_\lambda$ is indicated by (2.11), and $\gamma(m,n), h = h(x,y)$ are unit vector with two variants, namely

$$||h|| = \left\{ \int_0^\infty \int_0^\infty h^2 dxdy \right\}^{1/2} = 1, \qquad ||\gamma|| = \left(\sum_{m=1}^\infty \sum_{n=1}^\infty \gamma^2 \right)^{1/2} = 1$$

and $F^{p/2}, G^{q/2}, h$ are linearly independent. $\alpha^{p/2}, \beta^{q/2}, \gamma$ are also linearly independent.

Theorem 4. *Let* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 2 - min\{p,q\},\ A, B > 0, f(t), g(t) \geq 0$ *if*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < +\infty, 0 < \int_0^\infty t^{1-\lambda} g^q(t)dt < +\infty \text{ then}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda} \tag{3.3}$$

$$< \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\int_0^\infty x^{(1-\lambda)} f^p(x)dx \right)^{1/p} \left(\int_0^\infty x^{(1-\lambda)} g^q(x)dx \right)^{1/q} [1 - R(A,B,\lambda)]^k$$

where $R(A,B,\lambda) = (S_p(F,h) - S_q(G,h))^2$

$$h(x,y) = \frac{1}{(D_q(A,B)E)^{1/2}} \cdot \frac{e^{-x}}{(Ax+By)^{\lambda/2}} \left(\frac{x}{y} \right)^{(2-\lambda)/(2q)} \tag{3.4}$$

In particular. (i) for $\lambda = 1$ *we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)} dxdy \tag{3.5}$$

$$< \frac{\pi}{A^{1/q} B^{1/p} sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(x)dx \right)^{1/q} [1 - R(A,B,1)]^k$$

(ii) for $\lambda = 2$*, we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^2} dxdy \tag{3.6}$$

$$< \frac{1}{AB} \left(\int_0^\infty \frac{1}{x} f^p(x)dx \right)^{1/p} \left(\int_0^\infty \frac{1}{x} g^q(x)dx \right)^{1/q} [1 - R(A,B,2)]^k$$

Proof: By Lemma 2.1, we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda} dxdy = \int_0^\infty \int_0^\infty FG dxdy$$

$$\leq \left\{ \int_0^\infty \int_0^\infty F^p dxdy \right\}^{1/p} \left\{ \int_0^\infty \int_0^\infty G^q dxdy \right\}^{1/q} (1 - R(A,B,\lambda))^k \tag{3.7}$$

$$= \left\{ \int_0^\infty \omega_\lambda(A,B,q,x) f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty \omega_\lambda(B,A,p,y) g^q(y)dy \right\}^{1/q} (1 - R(A,B,\lambda))^k$$

$$= \left(\int_0^\infty \omega_\lambda(A,B,q,x) f^p(x)dx\right)^{1/p} \tag{3.8}$$

$$\times \left(\int_0^\infty \omega_\lambda(B,A,p,x) g^q(x)dx\right)^{1/q} (1 - R(A,B,\lambda))^k$$

In view of (2.7), we still have

$$\omega_\lambda(B,A,p,x) = k_\lambda(p)x^{1-\lambda} B^{1-\lambda-(2-\lambda)/p} A^{(2-\lambda)/p-1} \tag{3.9}$$

whence by (2.7), (3.8) and (3.9) we obtain (3.3).
It remains to discuss the expression of $R(A,B,\lambda)$. we may choose the function h indicated by (3.4).

$$||h|| = \left(\int_0^\infty \int_0^\infty h^2(x,y)dxdy\right)^{1/2}$$

$$= \left\{\frac{1}{D_q(A,B)E}\int_0^\infty e^{-2x}dx \int_0^\infty \frac{1}{(Ax+By)^\lambda}\left(\frac{x}{y}\right)^{(2-\lambda)/q} dy\right\}^{1/2}$$

$$= \left\{\frac{1}{D_q(A,B)E}\int_0^\infty e^{-2x}\omega_\lambda(A,B,q,x)dx\right\}^{1/2} = 1$$

According to Lemma 2.1 and the given h , we have

$$R(A,B,\lambda) = \left\{\left(\int_0^\infty \int_0^\infty F^{p/2} h dxdy\right)\left(\int_0^\infty \int_0^\infty F^p dxdy\right)^{-1/2} \right. \tag{3.10}$$

$$\left. - \left(\int_0^\infty \int_0^\infty G^{q/2} h dxdy\right)\left(\int_0^\infty \int_0^\infty G^q dxdy\right)^{-1/2}\right\}^2$$

Substituting (2.7),(3.1), (3.2), (3.4) and (3.9) into (3.10), we get

$$R(A,B,\lambda) = (S_p(F,h) - S_q(G,h))^2$$

In particular, for $A = B = \lambda = 1$, and $p = q = 2$, then $\Psi(y) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_0^\infty \frac{e^{-x}}{(x+y)}dx$

For $\quad 0 \leq 2 - min\{p,q\} < \lambda < 2, E = 2^{(\lambda-2)}\Gamma(2-\lambda)$
It is obvious that $F^{p/2}, G^{q/2}, h$ are linearly independent, so it is impossible for equality to hold in(3.7). Thus the proof of theorem is completed.

Remark 3.2 Clearly, the inequalities (3.3) (3.5) and (3.6) are the improvements of (2.1),(2.2) and (2.3) in [3] respectively.
Especially for $A = B = \lambda = 1$, by Theorem 4 we have.

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p} \tag{3.11}$$

$$\times \left(\int_0^\infty g^q(x)dx\right)^{1/q} [1 - R(1,1,1)]^k$$

when $p = q = 2$, the inequality (3.11) reduces to the inequality which is equivalent to the inequality (1.3) after simple computations. As a result, the inequalities (3.3), (3.5) and (3.11) are all the extensions of the inequality (1.3).

Theorem 5. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_1, b_1 > 0, a_n, b_n \geq 0 (n \in N)$ $2 - min\{p,q\} < \lambda \leq 2, A, B > 0$. *If* $0 < \sum_{n=1}^{\infty} n^{(1-\lambda)} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{(1-\lambda)} b_n^q < \infty$ *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^{\lambda}} < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left\{\sum_{n=1}^{\infty} n^{(1-\lambda)} a_n^p\right\}^{1/p} \times \left\{\sum_{n=1}^{\infty} n^{(1-\lambda)} b_n^q\right\}^{1/q} \left[1 - \overline{R}(A,B,\lambda)\right]^k \tag{3.12}$$

In particular. (i) for $\lambda = 1$ *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)} < \frac{\pi}{A^{1/q} B^{1/p} \, sin(\pi/p)} \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q} \left[1 - \overline{R}(A,B,1)\right]^k \tag{3.13}$$

(ii) for $\lambda = 2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^2} < \frac{1}{AB} \left(\sum_{n=1}^{\infty} \frac{1}{n} a_n^p\right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{1}{n} b_n^q\right)^{1/q} \left[1 - \overline{R}(A,B,2)\right]^k \tag{3.14}$$

(iii) for $2 - min\{p,q\} < \lambda \leq 2, A = B = 1$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < k_\lambda(p) \left(\sum_{n=1}^{\infty} n^{(1-\lambda)} a_n^p\right)^{1/p} \left(\sum_{n=1}^{\infty} n^{(1-\lambda)} b_n^q\right)^{1/q} \left[1 - \overline{R}(1,1,\lambda)\right]^k \tag{3.15}$$

where $\overline{R}(A,B,\lambda) = (S_p(\alpha,\gamma) - S_q(\beta,\gamma))^2$, *moreover the function* γ *is defined by*

$$\gamma = \begin{cases} 1 & m = n = 1 \\ 0 & m, n \in N \text{ but } m, n \text{ is not simultaneously equal to one} \end{cases} \tag{3.16}$$

Proof: By Lemma 2.2, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^{\lambda}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha \cdot \beta$$
$$\leq \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha^p\right\}^{1/p} \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta^q\right\}^{1/q} \left[1 - \overline{R}(A,B,\lambda)\right]^k$$
$$= \left\{\sum_{m=1}^{\infty} \overline{\omega}_\lambda(A,B,q,m) a_m^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} \overline{\omega}_\lambda(B,A,p,n) b_n^q\right\}^{1/q} (1 - \overline{R}(A,B,\lambda))^k$$
$$< \left\{\sum_{n=1}^{\infty} \omega_\lambda(A,B,q,n) a_n^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} \omega_\lambda(B,A,p,n) b_n^q\right\}^{1/q} (1 - \overline{R}(A,B,\lambda))^k \tag{3.17}$$

Substituting (2.7) and (3.9)into the inequality(3.17), it follows that the inequality (3.12) is valid.

Let us choose the function $\gamma(m,n)$ indicated by (3.16).
Obviously, $\|\gamma\|^2 = \sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\gamma^2 = 1$. It is easy to deduce that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^{p/2}\gamma = \frac{a_1^p}{(A+B)^{\lambda}} \quad \text{and} \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^{q/2}\gamma = \frac{b_1^q}{(A+B)^{\lambda}}.$$

According to Lemma 2.2 and $\gamma(m,n)$ selected, we have

$$\begin{aligned}\overline{R}(A,B,\lambda) = &\left\{\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^{p/2}\gamma\right)\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^{p}\right)^{-1/2}\right.\\ &\left.-\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^{q/2}\gamma\right)\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^{q}\right)^{-1/2}\right\}^2\\ &= \left(S_p(\alpha,\gamma) - S_q(\beta,\gamma)\right)^2\end{aligned}$$

Thus the theorem is proved.

Remark 3.4 Obviously, the inequality (3.12),(3.13) and (3.14) are the improvements of the inequalities (3.5),(3.6) and (3.7) in [3] respectively.

Corollary 3.5 Let f and g be real functions, $\lambda > 0, A, B > 0$. if $0 < \int_0^{\infty} t^{1-\lambda} f^2(t)dt < \infty, 0 < \int_0^{\infty} t^{1-\lambda} g^2(t)dt < \infty$ then

$$(3.18) \qquad \int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(Ax+By)^{\lambda}}dxdy$$
$$< \frac{1}{(AB)^{\lambda/2}}B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_0^{\infty} t^{(1-\lambda)}f^2(t)dt\int_0^{\infty} t^{(1-\lambda)}g^2(t)dt\right)^{1/2}[1-r(A,B,\lambda)]^{1/2}$$

Corollary 3.6 Let $0 < \lambda \le 2, a_1, b_1 > 0$, and $\{a_n\}, \{b_n\}$ be sequences of real numbers. if $A > 0, B > 0, 0 < \sum_{n=1}^{\infty} n^{(1-\lambda)}a_n^2 < \infty, \sum_{n=1}^{\infty} n^{(1-\lambda)}b_n^2 < \infty$ then

$$(3.19) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}}$$
$$< \frac{1}{(AB)^{\lambda/2}}B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{n=1}^{\infty} n^{(1-\lambda)}a_n^2\sum_{n=1}^{\infty} n^{(1-\lambda)}b_n^2\right\}^{1/2}[1-\overline{r}(A,B,\lambda)]^{1/2}$$

Remark 3.7 Clearly, the inequality (3.18) and (3.19) are improvements of the inequalities (2.6) and (3.1) respectively in [6]. Therefore, the inequalities (3.3) and (3.12) are the extensions of the inequalities (2.6) and (3.1) in [6] respectively.

Remark 3.8 Concerning the best coefficients, the results in the paper are not too conflict to ones in [3]. Since the best coefficients in [3] are constant, but the coefficients in this paper are dependent on $f(x), g(y)$ or a_m, b_n. The authors show respect for [3].

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