INTEGRAL REPRESENTATION OF MATHIEU $(\mathbf{a}, \boldsymbol{\lambda})$-SERIES

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Abstract. In the article an integral representation of Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series $S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda}) = \sum_{n=0}^{\infty} a(n)(\lambda(n)+\varrho)^{-p}$ is obtained generalizing certain results by Guo, Tomovski, Tomovski and Trenčevski, and Qi. Bilateral bounding inequalities are given for $S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda})$ using this integral expression.

1. Introduction and preparation

Concerning Mathieu series integral representations and bounding inequalities few open problems were posed by Feng Qi in his recent articles. As the most general ones could be mentioned [8, Open Problem 4.3.], [9, Open Problem 1.]. In this article we consider the series

$$S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda}) = \sum_{n=0}^{\infty} \frac{a(n)}{(\lambda(n)+\varrho)^p}, \qquad \varrho, p \in (0, \infty) \tag{1}$$

assuming that the sequences $\mathbf{a} := \{a(n)\}_{n \in \mathbb{N}_0}, \boldsymbol{\lambda} := \{\lambda(n)\}_{n \in \mathbb{N}_0}$ are positive, real and $\boldsymbol{\lambda}$ monotonously increases to ∞. The series (1) we call *Mathieu* $(\mathbf{a}, \boldsymbol{\lambda})$*-series* in the sequel. Obtaining a double integral representation of $S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda})$ we will derive upper and lower bounds for the Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series. Results obtained by Guo, Tomovski, Tomovski and Trenčevski, and Qi are generalized by our main results, compare [1], [6], [7], [8], [9] respectively; Yang's article is devoted to other problems around Mathieu series. At this point we have to point out that Cerone and Lenard give integral expression for the so-called *generalized Mathieu series* using first kind Bessel function, [2], [3].

Our main mathematical tools are the Laplace integral representation of general Dirichlet series and the Euler - McLaurin summation formula. The Laplace integral form of general Dirichlet series is

$$\sum_{n=0}^{\infty} a(n) e^{-\lambda(n)x} = x \int_0^{\infty} e^{-xt} A(t) dt, \tag{2}$$

where the sequence $\{\lambda(n)\}_{n \in \mathbb{N}_0}$ is monotonous increasing and $\lim_{n \to \infty} \lambda(n) = \infty$. Here

$$A(t) := \sum_{\lambda(n) \le t} a(n), \tag{3}$$

see e.g. [4, Part **C**, **V.5.1**].

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The second mathematical tool is the Euler-McLaurin summation formula

$$\sum_{j=0}^{n} a_j = \int_0^n a(u)du + \frac{1}{2}(a_n + a_0) + \int_0^n a'(u)B_1(u)du, \tag{4}$$

valid for $a_x = a(x) \in C^1[0,\infty)$, [5, **296**, p.539]. Here $B_1(u) = \{u\} - \frac{1}{2}$ is the Bernoulli polynomial of first degree, while $[u], \{u\} = u - [u]$ stand for the integer and the fractional part of u (so do in the whole manuscript).

2. Integral representation of Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series

The main result of this article is the integral representation of $S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda})$ for positive, real r, p and $\mathbf{a}, \boldsymbol{\lambda}$ defined already. At first we remark that

$$S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda}) = \frac{1}{\Gamma(p)} \int_0^\infty x^{p-1} e^{-\varrho x} \left(\sum_{n=0}^\infty a(n) e^{-\lambda(n)x} \right) dx. \tag{5}$$

The Dirichlet series $\sum_{n=0}^\infty a(n)e^{-\lambda(n)x}$ possesses Laplace integral form (2) where

$$A(t) = \sum_{n=0}^{[\lambda^{-1}(t)]} a(n) \tag{6}$$

and $[\lambda^{-1}(t)]$ is its *counting function*[1].

By Euler-McLaurin formula we deduce

$$\begin{aligned} A(t) &= \int_0^{[\lambda^{-1}(t)]} a(u)du + \frac{1}{2}(a(0) + a([\lambda^{-1}(t)])) + \int_0^{[\lambda^{-1}(t)]} a'(u)B_1(u)du \\ &= a(0) + \int_0^{[\lambda^{-1}(t)]} (a(u) + a'(u)\{u\})du. \end{aligned} \tag{7}$$

Now, by (4), (6) and (7) we easily get

$$\begin{aligned} S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda}) &= \frac{1}{\Gamma(p)} \int_0^\infty \int_0^\infty x^p e^{-(\varrho+t)x} A(t) dx dt \\ &= \frac{a(0)}{\varrho^p} + p \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \frac{a(u) + a'(u)\{u\}}{(\varrho+t)^{p+1}} dt du. \end{aligned} \tag{8}$$

This finishes the derivation of the integral representation.

Theorem 1. *Let $\varrho, p > 0$, let $a \in C^1[0,\infty)$ be nonnegative and let $\lambda(n)$ be positive real monotonuos increasing divergent sequence such that the Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$-series (1) converges. Then we have*

$$S(\varrho, p, \mathbf{a}, \boldsymbol{\lambda}) = \frac{a(0)}{\varrho^p} + p \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \frac{a(u) + a'(u)\{u\}}{(\varrho+t)^{p+1}} dt du. \tag{9}$$

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[1] λ^{-1} is the unique inverse of λ, consult e.g. [4, Part **C**, **V.1.1.**]. Indeed, because λ is monotonous increasing it has unique inverse which increases as well.

Remark 1. *The most useful form of the integral representation of $S(\varrho,p,\mathbf{a},\boldsymbol{\lambda})$ in deriving the bounding inequalities is* (9). *However, we can rearrange $A(t)$ into*

$$A(t)=a(0)+[\lambda^{-1}(t)]a([\lambda^{-1}(t)])-\int_0^{[\lambda^{-1}(t)]}a'(u)[u]du,$$

therefore (9) *becomes*

$$S(\varrho,p,\mathbf{a},\boldsymbol{\lambda})=\frac{a(0)}{\varrho^p}+p\int_0^\infty\frac{[\lambda^{-1}(t)]a([\lambda^{-1}(t)])-\int_0^{[\lambda^{-1}(t)]}a'(u)[u]du}{(\varrho+t)^{p+1}}\,dt.$$

Further calculation is senseless, it leads us back to (6) *and the Euler-McLaurin formula.*

3. Bounding inequalities for $S(\varrho,p,\mathbf{a},\boldsymbol{\lambda})$

In the remaining part of this note our main apparatus will be the previous integral. Since $0\le\{u\}<1$ by (9) we deduce

$$\begin{aligned}a(0)+\int_0^{[\lambda^{-1}(t)]}a(u)du\le A(t)&<a(0)+\int_0^{[\lambda^{-1}(t)]}(a(u)+a'(u))du\\&=a([\lambda^{-1}(t)])+\int_0^{[\lambda^{-1}(t)]}a(u)du.\end{aligned}\tag{10}$$

Applying these evaluations to (9) we deduce the following result.

Theorem 2. *Let the situation be the same as in the previous theorem. Then it holds true the bilateral inequality*

$$\frac{a(0)}{p\varrho^p}\le\frac{1}{p}S(\varrho,p,\mathbf{a},\boldsymbol{\lambda})-\int_0^\infty\int_0^{[\lambda^{-1}(t)]}\frac{a(u)}{(\varrho+t)^{p+1}}\,dtdu<\int_0^\infty\frac{a([\lambda^{-1}(t)])}{(\varrho+t)^{p+1}}\,dt.\tag{11}$$

Both bounding inequalities are sharp, i.e. cannot be improved.

Proof. It remains only the discussion of sharpness in (11). Indeed, (11) cannot be refined because in evaluation of Mathieu $(\mathbf{a},\boldsymbol{\lambda})$-series we apply only $0\le\{u\}<1$, which is sharp on the whole range of u. □

4. Discussion, special cases, corollaries

A. Putting $a(n)=2n^{\alpha/2},\ \lambda(n)=n^\alpha,\ \varrho=r^2,\ p+1\Rightarrow p$ into (1), we get the *generalized Mathieu type series*, reads as follows

$$S(r,p,\alpha)=\sum_{n=1}^\infty\frac{2n^{\alpha/2}}{(n^\alpha+r^2)^{p+1}},\qquad r,p,\alpha>0.\tag{12}$$

In the same time the right hand expression in (9) becomes

$$S(r,p,\alpha)=\frac{p+1}{\alpha+2}\int_0^\infty\frac{4[t^{1/\alpha}]^{\alpha/2+1}+\alpha(\alpha+2)\int_0^{[t^{1/\alpha}]}u^{\alpha/2-1}\{u\}du}{(r^2+t)^{p+2}}\,dt.\tag{13}$$

This integral representation gives an answer to Open Problem 4.3. posed in [8] which covers the Open Problem in [1] too. In the same time (13) precizes the integral expression for $S(r,p,\alpha)$ in [6, Theorem 1], therefore in [7] as well.

B. By A the bilateral bounding inequality (11) in our Theorem 2 becomes

$$0 \le S(r,p,\alpha) - \frac{4(p+1)}{\alpha+2}\int_1^\infty \frac{[t^{1/\alpha}]^{\alpha/2+1}}{(r^2+t)^{p+2}}\,dt < 2(p+1)\int_1^\infty \frac{[t^{1/\alpha}]^{\alpha/2}}{(r^2+t)^{p+2}}\,dt. \tag{14}$$

It is not hard to see that the bounds in (14) cannot be improved. The approach by Tomovski, who give bilateral bounds for the generalized Mathieu series $S(r,p,\alpha)$, was to apply the trapezoidal rule. We use the Euler-McLaurin summation formula and achieve a refinement of his bounds. However, suitable approximations of $[t^{1/\alpha}]$ in (14) lead to appropriate bounds, wich ones will be similar to Tomovski's results, see [6], [7].

C. The generalized Mathieu series $S(r,p,2)$ is considered by Cerone and Lenard in different framework. They proved that

$$S(r,p,2) = \frac{\sqrt{\pi}}{(2r)^{p-1/2}\Gamma(p+1)}\int_0^\infty \frac{t^{p+1/2}}{e^t-1}J_{p-1/2}(rt)dt, \tag{15}$$

where $J_\nu(x)$ is the $\nu^{\rm th}$ order Bessel function of first kind, [2, Theorem 1], [3, Theorem 2.1].

Comparing (13) for $\alpha=2$ and (15) we deduce the unexpected and surprising equality

$$\int_0^\infty \frac{t^{p+1/2}J_{p-1/2}(rt)}{e^t-1}\,dt = \frac{\Gamma(p+2)(2r)^{p-1/2}}{\sqrt{\pi}}\int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{p+2}}\,dt. \tag{16}$$

D. Finally, let $\mathbf{a}=\{a_n\}_{n\in\mathbb{N}_0},\ a_0=0$ be a positive sequence. Specifying $a(n)=a_n^\beta,\ \lambda(n)=a_n^\alpha,\ \varrho=r^2$ in the Mathieu $(\mathbf{a},\boldsymbol{\lambda})$-series (1), we get the generalized Mathieu series

$$S(r,p,\alpha,\beta,\mathbf{a}) = \sum_{n=1}^\infty \frac{a_n^\beta}{(a_n^\alpha+r^2)^p}, \tag{17}$$

proposed as the subject of consideration by Qi, [9, Open Problem 1]. Now, we are ready to give some answers to his questions.

1. The series (17) converges when $\mathbf{a}$ is monotonous increasing, a_n^{-1} vanishes with sufficient convergence rate and $\alpha p-\beta>0$.

2. Assume $a_x=a(x)\in C^1[0,\infty), a'(x)>0$ and let a^{-1} be the inverse of a. The integral expression for the series (17) which we deduce from (9) is

$$S(r,p,\alpha,\beta,\mathbf{a}) = p\int_0^\infty\int_0^{[a^{-1}(t^{1/\alpha})]} \frac{a^{\beta-1}(u)(a(u)+\beta a'(u)\{u\})}{(t+r^2)^{p+1}}\,dtdu. \tag{18}$$

3. Under previous assumptions it holds

$$0 \le \frac{1}{p}S(r,p,\alpha,\beta,\mathbf{a}) - \int_0^\infty\int_0^{[a^{-1}(t^{1/\alpha})]} \frac{a^\beta(u)}{(t+r^2)^{p+1}}\,dtdu < \int_0^\infty \frac{a^\beta([a^{-1}(t^{1/\alpha})])}{(t+r^2)^{p+1}}\,dt, \tag{19}$$

and these bounds are sharp.

Finally, it has to mentioned that F.Qi's Open Problem 1, [9] is not fully covered by these results, since he was interested in the convergence of $S(r, p, \alpha, \beta, \mathbf{a})$ in the case of general positive $\mathbf{a}$, while concerning this **1.** gives only sufficient conditions.

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