THE BEST CONSTANT FOR AN INEQUALITY

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Abstract. In this short note, we prove an algebraic inequalities:

$$\frac{1}{x^2} - \frac{1}{12} < \frac{e^{-x}}{(1-e^{-x})^2},$$

where $0 < x < 1$, and the constant $\frac{1}{12}$ be the best possible.

1. Introduction and Main Results

There is an interesting open problem in [1], that is:

For $0 < x < 1$, then there is a positive number c which make the following inequalities

$$\frac{1}{x^2} - c < \frac{e^{-x}}{(1-e^{-x})^2} < \frac{1}{x^2} \tag{1.1}$$

holding, and find the best constant c for (1.1).

The right inequality of (1.1) is easily proved, i.e.

Proof. The right inequality of (1.1) is equivalent to

$$(1-e^{-x})^2 - x^2e^{-x} > 0. \tag{1.2}$$

Define the function

$$f(x) = (1-e^{-x})^2 - x^2e^{-x}, x \in (0,1), \tag{1.3}$$

we have

$$f'(x) = e^{-2x}(x^2e^x - 2xe^x + 2e^x - 2). \tag{1.4}$$

Setting

$$g(x) = x^2e^x - 2xe^x + 2e^x - 2,\ x \in (0,1), \tag{1.5}$$

and calculating the derivative for $g(x)$, we get $g'(x) = x^2e^x$. It is obvious that $g'(x) > 0$ for all real numbers, and implies that the function g is strictly monotone increasing on interval (0, 1). So $g(x) > g(0) = 0$ for $0 < x < 1$. And the same time, we know $f'(x) > 0$ for $0 < x < 1$, and the function f is strictly monotone increasing ones on interval $(0,1)$, too. Therefore, $f(x) > f(0) = 0$ for $0 < x < 1$. The proof is completed. ∎

Also, our task is finding the best constant c and proving it for the left inequality of (1.1). In this short note, we will prove the following theorem.

Theorem 1.1. *Let $0 < x < 1$, then the inequality*

$$\frac{1}{x^2} - \frac{1}{12} < \frac{e^{-x}}{(1-e^{-x})^2} \tag{1.6}$$

holds, and the constant $\frac{1}{12}$ be the best possible.

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2. Lemma

In order to prove Theorem1.1 below, we require the following two lemmas.

Lemma 2.1. *If n be a positive integer, then we have*

$$(n+2)\cdot 2^{n+1} \geqslant 5n^2+5n+2. \tag{2.1}$$

Proof. Using the mathematical induction.

(i) When $n=1$, it is obvious that (2.1) holds.

(ii) Suppose (2.1) holds when $n=k$ ($k \geqslant 1$), then

$$(k+2)\cdot 2^{k+1} \geqslant 5k^2+5k+2. \tag{2.2}$$

While $n=k+1$, (2.1) is equivalent to

$$(k+3)\cdot 2^{k+2} \geqslant 5(k+1)^2+5(k+1)+2. \tag{2.3}$$

From (2.2), we only need prove

$$2(5k^2+5k+2)+2^{k+2} \geqslant 5(k+1)^2+5(k+1)+2, \tag{2.4}$$

that is equivalent to

$$5(k^2-k)+2^{k+2}-8 \geqslant 0. \tag{2.5}$$

Therefore, we have (2.5), because k be a natural number. Thus, inequality (2.1) holds for all k be a natural number. ∎

Lemma 2.2. *If $x>0$, then we have*

$$12+x-x^2-12e^x+10xe^x+xe^{2x}-5x^2e^x>0. \tag{2.6}$$

Proof. Utilizing the fact that

$$e^x = 1+\sum_{k=1}^{\infty}\frac{x^k}{k!}, \tag{2.7}$$

inequality (2.6) is equivalent to

$$\sum_{k=1}^{\infty}\left[\frac{(k+2)\cdot 2^{k+1}-5k^2-5k-2}{(k+2)!}\right]x^{k+2}>0. \tag{2.8}$$

Using Lemma2.1, combining $x>0$, we can conclude that Lemma2.2 is correct. The proof of Lemma2.2 is completed. ∎

3. The Proof of Theorem1.1

Proof. It is obvious that inequality (1.6) is equivalent to

$$x^2e^{-x}-(1-\frac{1}{12}x^2)(1-e^{-x})^2>0. \tag{3.1}$$

Define the function

$$f(x)=x^2e^{-x}-(1-\frac{1}{12}x^2)(1-e^{-x})^2, x\in(0,1). \tag{3.2}$$

Calculating the derivative for $f(x)$, we get

$$f'(x)=\frac{1}{6}e^{-2x}(12+x-x^2-12e^x+10xe^x+xe^{2x}-5x^2e^x). \tag{3.3}$$

Using Lemma2.2, we find that $f'(x)>0$. Thus f is strictly monotone increasing function on interval $(0,1)$. Obviously, $f(0)=0$. So $f(x)>0$ for $0<x<1$.

Next,we prove that $c=\frac{1}{12}$ is the best constant for inequality (1.6).

Suppose inequality (1.6) holds for any $0 < x < 1$. Applying Taylor's Theorem to the functions x^2e^{-x} and $(1-cx^2)(1-e^{-x})^2$, we obtain

$$x^2e^{-x} = x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \cdots, \tag{3.4}$$

and

$$(1-cx^2)(1-e^{-x})^2 = x^2 - x^3 + (\frac{7}{12} - c)x^4 + (c - \frac{1}{4})x^5 + \cdots. \tag{3.5}$$

With simple manipulations (3.4) and (3.5), together with $x^2e^{-x} > (1-cx^2)(1-e^{-x})^2$, yield

$$\frac{1}{2} \geqslant \frac{7}{12} - c \tag{3.6}$$

From (3.6), it immediately follows that $c \geqslant \frac{1}{12}$. So $c = \frac{1}{12}$ is the best constant for inequality (1.6). Theorem1.1 is proved. ∎

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