SOME REMARKS ON THE NOISELESS CODING THEOREM

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Abstract. An improvement of the Noiseless Coding Theorem for certain probability distributions is given.

1. Introduction

The following analytic inequality for the $\log(\cdot)$ map is well known in the literature (see for example [1, Lemma 1.2.2, p. 22]):

Lemma 1. *Let $P = (p_1, ..., p_n)$ be a probability distribution that is, $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$. Let $Q = (q_1, ..., q_n)$ have the property that $0 \leq q_i \leq 1$ and $\sum_{i=1}^{n} q_i \leq 1$, then*

$$\sum_{i=1}^{n} p_i \log_b \frac{1}{p_i} \leq \sum_{i=1}^{n} p_i \log_b \frac{1}{q_i} \quad (b > 1) \tag{1.1}$$

where $0 \log_b \frac{1}{0} = 0$ and $p \log_b \frac{1}{0} = +\infty$ for $p > 0$. Furthermore, the equality holds if and only if $q_i = p_i$ for all i.

Note that the proof of this result in [1] uses the elementary inequality:

$$\ln x \leq x - 1 \quad \text{for all } x > 0.$$

We give here an alternative proof based on the concavity of the mapping $\log_r(\cdot)$.

As the mapping $f(x) = \log_r(x)$ $(r > 1)$ is a strictly concave mapping on $(0, \infty)$, we have

$$f(x) - f(y) \geq f'(x)(x - y)$$

for all $x, y > 0$, i.e., as $f'(x) = \frac{1}{\ln r} \cdot \frac{1}{x}$ for $x > 0$,

$$\log_r x - \log_r y \geq \frac{1}{\ln r}\left(\frac{x - y}{x}\right) \tag{1.2}$$

for all $x, y > 0$.

Choosing $x = \frac{1}{q_i}, y = \frac{1}{p_i}$, in (1.2) gives

$$\log_r \frac{1}{q_i} - \log_r \frac{1}{p_i} \geq \frac{1}{\ln r}\left(\frac{p_i - q_i}{p_i}\right) \tag{1.3}$$

for all $i \in \{1, ..., n\}$.

Multiplying this inequality by $p_i > 0$ $(i = 1, ..., n)$ we get

$$p_i \log_r \frac{1}{q_i} - p_i \log_r \frac{1}{p_i} \geq \frac{1}{\ln r}(p_i - q_i)$$

Date: February 26, 2004.
1991 *Mathematics Subject Classification.* Primary 94 A xx; Secondary 26 D xx.
Key words and phrases. Entropy, Instantaneous Codes, The Noiseless Coding Theorem.

for all $i \in \{1, ..., n\}$.

Summing over i from 1 to n, gives

$$\sum_{i=1}^{n} p_i \log_r \frac{1}{q_i} - \sum_{i=1}^{n} p_i \log_r \frac{1}{p_i} \geq \frac{1}{\ln r}\left(\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i\right)$$

$$= \frac{1}{\ln r}\left(1 - \sum_{i=1}^{n} q_i\right) \geq 0$$

and the inequality (1.1) is obtained.

The case of equality follows by the strict concavity of the mapping log_r.

In this paper, by use of (1.1), we point out an improvement to the Noiseless Coding Theorem.

2. The Results

Consider an encoding scheme $(c_1, ..., c_n)$ for a probability distribution $(p_1, ..., p_n)$. The *average codeword length of an encoding scheme* $(c_1, ..., c_n)$ for $(p_1, ..., p_n)$ is

$$AveLen(c_1, ..., c_n) = \sum_{i=1}^{n} p_i len(c_i).$$

We denote the length $len(c_i)$ by l_i.

The $r-$ary entropy of a probability distribution is given by

$$H_r(c_1, ..., c_n) = \sum_{i=1}^{n} p_i \log_r\left(\frac{1}{p_i}\right).$$

The following theorem is well known in the literature (see for example [1, Theorem 2.3.1, p. 62]):

Theorem 2. *Let $C = (c_1, ..., c_n)$ be an instantaneous (or uniquely decipherable) encoding scheme for $P = (p_1, ..., p_n)$, then,*

$$H_r(p_1, ..., p_n) \leq AveLen(c_1, ..., c_n)$$

with equality if and only if $l_i = \log_r\left(\frac{1}{p_i}\right)$ for all $i = 1, ..., n$.

The following result, providing a counterpart inequality, holds.

Theorem 3. *Let $P = (p_1, ..., p_n)$ be a given probability distribution and $r \in \mathbf{N}, r \geq 2$. If $\varepsilon > 0$ is fixed and there exists natural numbers $l_1, ..., l_n$ such that:*

$$\log_r\left(\frac{1}{p_i}\right) \leq l_i \leq \log_r\left(\frac{r^{\varepsilon}}{p_i}\right) \tag{2.1}$$

for all $i \in \{1, ..., n\}$, then there exists an instantaneous $r-$ary code $C = (c_1, ..., c_n)$ with codeword length $len(c_i) = l_i$ such that

$$H_r(p_1, ..., p_n) \leq AveLen(c_1, ..., c_n) \leq H_r(p_1, ..., p_n) + \varepsilon. \tag{2.2}$$

Proof. Note that (2.1) is equivalent to

$$\frac{1}{p_i} \leq r^{l_i} \leq \frac{r^{\varepsilon}}{p_i} \quad \text{for all } i \in \{1, ..., n\}. \tag{2.3}$$

Now, since $\frac{1}{r^{l_i}} \leq p_i \ (i = 1, ..., n)$, it follows that

$$\sum_{i=1}^{n} \frac{1}{r^{l_i}} \leq \sum_{i=1}^{n} p_i = 1$$

and by Kraft's theorem (see for example [1, Theorem 2.1.2, p. 44]), there exists an instantaneous $r-$ary code $C = (c_1, ..., c_n)$ such that $len\,(c_i) = l_i$.

Obviously, by Theorem 2, the first inequality in (2.2) holds.

We have:

$$AveLen\,(c_1, ..., c_n)$$
$$= \sum_{i=1}^{n} p_i l_i = \sum_{i=1}^{n} p_i \log_r r^{l_i} = \sum_{i=1}^{n} p_i \log_r \frac{1}{q_i}$$

choosing $q_i = \frac{1}{r^{l_i}} \in [0, 1]$. Also, by Kraft's theorem, $\sum_{i=1}^{n} q_i \leq 1$.

By Lemma 1, we have,

$$0 \leq \sum_{i=1}^{n} p_i \log_r \frac{1}{q_i} - \sum_{i=1}^{n} p_i \log_r \frac{1}{p_i} = AveLen\,(c_1, ..., c_n) - H_r\,(p_1, ..., p_n)$$
$$= \sum_{i=1}^{n} p_i \left(\log_r r^{l_i} - \log_r \frac{1}{p_i} \right) = \left| \sum_{i=1}^{n} p_i \left(\log_r r^{l_i} - \log_r \frac{1}{p_i} \right) \right|$$
$$\leq \sum_{i=1}^{n} p_i \left| l_i - \log_r \left(\frac{1}{p_i} \right) \right| \leq \varepsilon \sum_{i=1}^{n} p_i = \varepsilon$$

since, by (2.1), $0 \leq l_i - \log_r \frac{1}{p_i} \leq \log_r r^{\varepsilon} = \varepsilon$. ∎

We shall use the notation:

$$MinAveLen_r\,(p_1, ..., p_n)$$

to denote the minimum average codeword length among all $r-$ary instantaneous encoding schemes for the probability distribution $P = (p_1, ..., p_n)$.

The following Noiseless Coding Theorem is well known in the literature (see for example [1, Theorem 2.3.2, p. 64]):

Theorem 4. *For any probability distribution $P = (p_1, ..., p_n)$ we have:*

$$H_r\,(p_1, ..., p_n) \leq MinAveLen_r\,(p_1, ..., p_n) < H_r\,(p_1, ..., p_n) + 1. \tag{2.4}$$

The following question is then a natural one to pose.

Question: *Is it possible to replace the constant 1 in the above inequality by a smaller one $\varepsilon \in (0, 1)$ and, if so, under what conditions for the probability distribution $P = (p_1, ..., p_n)$?*

The following is a partial answer to this question:

Theorem 5. *Let r be a given natural number and $\varepsilon \in (0, 1)$. If a probability distribution $P = (p_1, ..., p_n)$ satisfies the condition that every closed interval of real numbers*

$$I_i = \left[\log_r \left(\frac{1}{p_i} \right), \log_r \left(\frac{r^{\varepsilon}}{p_i} \right) \right], \qquad i \in \{1, ..., n\},$$

contains one natural number, then, for that probability distribution P, we have:

$$H_r(p_1,...,p_n) \leq MinAveLen_r(p_1,...,p_n) \leq H_r(p_1,...,p_n) + \varepsilon. \tag{2.5}$$

Proof. Suppose that $l_i \in I_i$ $(i = 1,...,n)$ are these natural numbers, then, as above,

$$\sum_{i=1}^{n} \frac{1}{r^{l_i}} \leq \sum_{i=1}^{n} p_i = 1$$

and by Kraft's theorem there exists an instantaneous code $C = (c_1,...,c_n)$ such that $len(c_i) = l_i$. For this code we have (2.1) and, by Theorem 3, the inequality (2.2) for C. Taking the infimum in this inequality over all $r-$ary instantaneous codes, gives (2.5). ∎

Remark 1. *The lengths of the intervals I_i are,*

$$len(I_i) = \log_r \left(\frac{r^\varepsilon}{p_i}\right) - \log_r \frac{1}{p_i} = \varepsilon \in (0,1), \qquad i = 0,...,n$$

but we cannot be sure that I_i always contains a natural number. Also, I_i could contain at most one natural number.

The following result can be useful in practice.

Practical Criterion. *Let a_i be n natural numbers, $i = 1,...,n$. If p_i $(i = 1,...,n)$ are such that*

$$\frac{1}{r^{a_i}} \leq p_i \leq \frac{r^\varepsilon}{r^{a_i}} \qquad \text{for } i = 1,...,n \tag{2.6}$$

and $\sum_{i=1}^{n} p_i = 1$, then there exists an instantaneous code $C = (c_1,...,c_n)$ with $len(c_i) = a_i$ $(i = 1,...,n)$ such that (2.2) holds for the probability distribution $P = (p_1,...,p_n)$.

For other recent results in the applications of Theory of Inequalities in Information Theory and Coding, see the following references.

References

[1] S. Roman, *Coding and Information Theory,* Springer-Verlag, New York, Berlin, Heidelberg, 1992.

[2] N.M. Dragomir and S.S Dragomir, An inequality for logarithms and its application in coding theory, *Indian J. Math.*, **43**(1) (2001), 13-20.

[3] S. S. Dragomir and C. J. Goh, Some bounds on entropy measures in information theory, *Appl. Math. Lett.,* **10**(3) (1997) 23–28.

[4] S. S. Dragomir and C. J. Goh, A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory, *Math. Comput. Modelling,* **24**(2) (1996) 1–11.

[5] M. Matić, C. E. M. Pearce and J. Pečarić, Improvements of some bounds on entropy measures in information theory, *Math. Inequal. and Appl.*, **1**(1998) 295–304.

[7] S.S. Dragomir, N.M. Dragomir and K. Pranesh, Some Estimations of Kraft number and related results, *Soochow Journal of Mathematics*, **24**(4)(1998), 291-296.

[8] S.S. Dragomir, M. Sholz and J. Sunde, Some upper bounds for relative entropy and applications, *Comp. & Math. with Applications*, **39**(2000), 91-100.

[10] C. Calude and C. Grozea, Kraft-Chaitin inequality revisited, *J. UCS*, **5**(1996), 306-310.

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