SOME INEQUALITIES FOR THE TRIANGLE INVOLVING FIBONACCI NUMBERS

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ABSTRACT. In this note classical inequalities and Fibonacci numbers are used to obtain some miscellaneous inequalities involving the elements of a triangle.

1. Introduction

The elements of a triangle are a source of many nice identities and inequalities. A similar interpretation exists for Fibonacci numbers. Many of these identities and inequalities have been documented in extensive lists that appear in the work of Botema [1], Mitrinovic [3] and Koshy [2]. However, as far as we know, miscellaneous geometric inequalities for the elements of a triangle involving Fibonacci numbers never have appeared. In this paper, using classical inequalities and Fibonacci numbers some of these inequalities are given.

2. The Inequalities

In what follows some inequalities for the triangle are stated and proved. We start with

Theorem 2.1. *In all triangle $\triangle ABC$, with the usual notations, the following inequality*

$$a^2F_n + b^2F_{n+1} + c^2F_{n+2} \geq 4S\sqrt{F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n} \tag{2.1}$$

holds.

Proof. Let us denote by $k = 4\sqrt{F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n}$. Then, (2.1) reads

$$a^2F_n + b^2F_{n+1} + c^2F_{n+2} \geq kS. \tag{2.2}$$

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Taking into account the Cosine Law, we have

$$a^2F_n + b^2F_{n+1} + (a^2 + b^2 - 2ab\cos C)F_{n+2} \geq \frac{1}{2}kab\sin C,$$

or equivalently,

$$2\,\frac{a}{b}\Big(F_n + F_{n+2}\Big) + 2\,\frac{b}{a}\Big(F_{n+1} + F_{n+2}\Big) - \Big(4F_{n+2}\cos C + k\sin C\Big) \geq 0.$$

From Cauchy-Buniakovski-Schwarz's inequality applied to $(4F_{n+2}, k)$ and $(\cos C, \sin C)$, we obtain

$$4F_{n+2}\cos C + k\sin C \leq \sqrt{16F_{n+2}^2 + k^2}.$$

On the other hand, from AM-GM inequality, we get

$$2\frac{a}{b}(F_n + F_{n+2}) + 2\frac{b}{a}(F_{n+1} + F_{n+2}) \geq 4\sqrt{(F_n + F_{n+2})(F_{n+1} + F_{n+2})}.$$

Taking into account the preceding inequalities, we have

$$\begin{aligned} &2\frac{a}{b}(F_n + F_{n+2}) + 2\frac{b}{a}(F_{n+1} + F_{n+2}) - (4F_{n+2}\cos C + k\sin C) \\ &\geq 4\sqrt{(F_n + F_{n+2})(F_{n+1} + F_{n+2})} - (4F_{n+2}\cos C + k\sin C) \\ &\geq 4\sqrt{(F_n + F_{n+2})(F_{n+1} + F_{n+2})} - \sqrt{16F_{n+2}^2 + k^2} \geq 0 \end{aligned}$$

when $k \leq 4\sqrt{F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n}$.

Consequently, (2.1) holds and Theorem 1 is proved. □

As an immediate consequence of the preceding result we obtain the following inequality.

Corollary 2.2. *In all triangle $\triangle ABC$, holds*

$$a^2F_n + b^2F_{n+1} + c^2F_{n+2} \geq 4S\left(\sum_{k=1}^{n+2} F_k^2 - F_{n+1}^2\right)^{1/2}. \tag{2.3}$$

Proof. In fact, from Theorem 1 we have

$$\begin{aligned} a^2F_n + b^2F_{n+1} + c^2F_{n+2} &\geq 4S\sqrt{F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n} \\ &= 4S\sqrt{F_nF_{n+1} + F_{n+2}^2} \\ &= 4S\sqrt{F_1^2 + F_2^2 + \ldots + F_n^2 + F_{n+2}^2}. \end{aligned}$$

Note that in the last expression we have used the fact that $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$. Therefore,

$$a^2 F_n + b^2 F_{n+1} + c^2 F_{n+2} \geq 4S \left(\sum_{k=1}^{n+2} F_k^2 - F_{n+1}^2 \right)^{1/2}$$

and the proof is complete. □

Before stating our next result we give a Lemma that we will use further on.

Lemma 2.3. *Let x, y, z and a, b, c be strictly positive real numbers. Then, holds*

$$3\left(yza^2 + zxb^2 + xyc^2\right) \geq \left(a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy}\right)^2.$$

Proof. Let $\overrightarrow{u} = \left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right)$ and $\overrightarrow{v} = (a, b, c)$. By applying Cauchy-Buniakovski-Schwarz's inequality, we get

$$\left[\left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right) \cdot (a, b, c)\right]^2 \leq \| \left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right) \|^2 \| (a, b, c) \|^2$$

or equivalently,

$$\left(a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy}\right)^2 \leq (yz + zx + xy)(a^2 + b^2 + c^2). \tag{2.4}$$

On the other hand, by applying the rearrangement inequality yields

$$a^2yz + b^2zx + c^2xy \geq b^2yz + c^2zx + a^2xy,$$
$$a^2yz + b^2zx + c^2xy \geq b^2xy + a^2zx + c^2yz.$$

Hence, the right hand side of (2.4) becomes

$$(yz + zx + xy)(a^2 + b^2 + c^2) \leq 3(yza^2 + zxb^2 + xyc^2)$$

and the proof is complete.

□

In particular, setting $x = F_n, y = F_{n+1}$, and $z = F_{n+2}$ in the preceding Lemma, we get the following

Theorem 2.4. *If a, b and c are the sides of triangle $\triangle ABC$, then*

$$3\left(F_{n+1}F_{n+2}\, a^2 + F_{n+2}F_n\, b^2 + F_nF_{n+1}\, c^2\right)$$
$$\geq \left(a\sqrt{F_{n+1}F_{n+2}} + b\sqrt{F_{n+2}F_n} + c\sqrt{F_nF_{n+1}}\right)^2.$$

Finally, we will use the preceding result to state and prove the following

Theorem 2.5. *Let $\triangle ABC$ be a triangle, then for $\alpha \in \left[0, \frac{\pi}{2}\right)$, we have*

$$\sqrt{F_{n+1}F_{n+2}}\cos(C-\alpha)+\sqrt{F_{n+2}F_n}\cos(B-\alpha)+\sqrt{F_nF_{n+1}}\cos(A-\alpha) \leq 2F_{n+2}\cos\left(\frac{\pi}{3}-\alpha\right).$$

Proof. By applying Botema inequality [1] and Theorem 2, we get

$$\left(a\sqrt{F_{n+1}F_{n+2}}+b\sqrt{F_{n+2}F_n}+c\sqrt{F_nF_{n+1}}\right)^2$$

$$\leq 3\left(F_{n+1}F_{n+2}\,a^2+F_{n+2}F_n\,b^2+F_nF_{n+1}\,c^2\right) \leq 3R^2(F_n+F_{n+1}+F_{n+2})^2,$$

and from it,

$$a\sqrt{F_{n+1}F_{n+2}}+b\sqrt{F_{n+2}F_n}+c\sqrt{F_nF_{n+1}} \leq R\sqrt{3}(F_n+F_{n+1}+F_{n+2}).$$

Since $a = 2R\sin A, b = 2R\sin B$, and $c = 2R\sin C$, then from the preceding inequality, we obtain

$$\sqrt{F_{n+1}F_{n+2}}\,\sin A+\sqrt{F_{n+2}F_n}\,\sin B+\sqrt{F_nF_{n+1}}\,\sin C \leq \sqrt{3}F_{n+2}. \tag{2.5}$$

On the other hand, by applying the asymmetric trigonometric inequality of J. Wolstenholme ([3],[4]), we have

$$\sqrt{F_{n+1}F_{n+2}}\,\cos A+\sqrt{F_{n+2}F_n}\,\cos B+\sqrt{F_nF_{n+1}}\,\cos C \leq F_{n+2}. \tag{2.6}$$

Multiplying (2.5) by $\tan\alpha$, adding it up to (2.6), and after simplification, yields

$$\begin{aligned}
&\sqrt{F_{n+1}F_{n+2}}\,[\cos A\cos\alpha+\sin A\sin\alpha]\\
&+\sqrt{F_{n+2}F_n}\,[\cos B\cos\alpha+\sin B\sin\alpha]\\
&+\sqrt{F_nF_{n+1}}\,[\cos C\cos\alpha+\sin C\sin\alpha]\\
&\leq 2F_{n+2}\left[\cos\frac{\pi}{3}\cos\alpha+\sin\frac{\pi}{3}\sin\alpha\right]
\end{aligned}$$

and the proof is complete. □

References

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