CONVEXITY OF WEIGHTED EXTENDED MEAN VALUES

ALFRED WITKOWSKI

Abstract. We investigate convexity properties of the one-parameter families of Weighted Extended Mean Values

$$F_h(r) = F_h(r; a, b; x, y) = E(r, r+h; ax, by)/E(r, r+h; a, b)$$

where E is the Stolarsky mean and show that for arbitrary r_0 one of the inequalities

$$F_h(r_0+t)F_h(r_0-t) \leq (\geq) F_h^2(r_0)$$

holds for all real t. This implies some inequalities between classical means.

1. Introduction

Extended mean values of positive numbers x, y introduced by Stolarsky in [4] are defined as

$$E(r,s;x,y) = \begin{cases} \left(\frac{r}{s}\frac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)} & sr(s-r)(x-y) \neq 0, \\ \left(\frac{1}{r}\frac{y^r-x^r}{\log y - \log x}\right)^{1/r} & r(x-y) \neq 0,\ s=0, \\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r-x^r)} & r=s,\ r(x-y)\neq 0, \\ \sqrt{xy} & r=s=0,\ x-y\neq 0, \\ x & x=y. \end{cases} \tag{1}$$

It was shown in many ways that E increases in all variables (see [4, 3, 5, 6]). Alzer in [1] investigated the one-parameter mean

$$F(r) = F(r; x, y) = E(r, r+1; x, y) \tag{2}$$

and proved that for $x \neq y$ F is strictly log-convex for $r < -1/2$ and strictly log-concave for $r > -1/2$. He also proved that $F(r)F(-r) \leq F^2(0)$. In [2] Alzer obtained similar result for the Lehmers means

$$L(r) = L(r; x, y) = (x^{r+1} + y^{r+1})/(x^r + y^r). \tag{3}$$

In the present paper we generalize the above results.

In [7] we extended the Stolarsky means to a four-parameter family of means by adding positive weights a, b:

$$F(r,s;a,b;x,y) = \left(\frac{(ax)^s-(by)^s}{a^s-b^s}\Big/\frac{(ax)^r-(by)^r}{a^r-b^r}\right)^{1/(s-r)} \tag{4}$$

Date: 1 May 2004.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Extended mean values, mean, convexity.

Writing (4) as

$$F(r,s;a,b;x,y)=\frac{E(r,s;ax,by)}{E(r,s;a,b)}, \tag{5}$$

we see that F is continuous on $\mathbb{R}^2\times\mathbb{R}^2_+\times\mathbb{R}^2_+$. Obviously

$$F(r,s;a,a;x,y)=E(r,s;x,y).$$

The following monotonicity properties of F have been established in [7]:

Property 1. *F increases in x and y.*

Property 2. *F increases in r and s if $(x-y)(a^2x-b^2y)>0$ and decreases if $(x-y)(a^2x-b^2y)<0$*

Property 3. *F increases in a if $(x-y)(r+s)>0$ and decreases if $(x-y)(r+s)<0$, F decreases in b if $(x-y)(r+s)>0$ and increases if $(x-y)(r+s)<0$*

2. Main result

In this paper we investigate convexity properties of one-parameter means defined as

$$F_h(r)=F_h(r;a,b;x,y)=F(r,r+h;a,b;x,y). \tag{6}$$

It is obvious that the monotonicity of F_h matches that of F. The main result consists of the following:

Theorem 1. *If $(x-y)(a^2x-b^2y)>0$ then $F_h(r)$ is log-convex in $(-\infty,-h/2)$ and log-concave in $(-h/2,\infty)$.*

If $(x-y)(a^2x-b^2y)<0$ then F_h is log-concave in $(-\infty,-h/2)$ and log-convex in $(-h/2,\infty)$.

Theorem 2. *If F_h is log-convex in a neighborhood of r_0 then for every real t*

$$F_h(r_0-t)F_h(r_0+t)\geq F_h^2(r_0).$$

If F_h is log-concave in a neighborhood of r_0 then for every real t

$$F_h(r_0-t)F_h(r_0+t)\leq F_h^2(r_0).$$

The following corollaries are immediate consequence of theorems 1 and 2:

Corollary 3. *For $x\neq y$ the one-parameter mean $F(r)$ is log-convex for $r<-1/2$ and log-concave for $r>-1/2$. If $r_0>-1/2$ then for all real t $F(r_0-t)F(r_0+t)\leq F^2(r_0)$. For $r_0<-1/2$ the inequality reverses.*

Proof. $F(r;x,y)=F_1(r;1,1;x,y)$. □

Corollary 4. *For $x\neq y$ the Lehmer mean $L(r)$ is log-convex for $r<-1/2$ and log-concave for $r>-1/2$. If $r_0>-1/2$ then for all real t $L(r_0-t)L(r_0+t)\leq L^2(r_0)$. For $r_0<-1/2$ the inequality reverses.*

Proof. $L(r;x,y)=F_1(r;x,y;x,y)$. □

In the last section we present some inequalities between classical means that can be obtained from theorem 2.

3. Lemmas

Note that in general case (6) can be written as

$$F_h(r) = y\left(\frac{A^{r+h}-1}{B^{r+h}-1}\bigg/\frac{A^r-1}{B^r-1}\right)^{1/h}, \tag{7}$$

where

$$A = \frac{ax}{by} \quad \text{and} \quad B = \frac{a}{b}.$$

It is enough to prove our theorems only in case the expression (7) makes sense. Other cases follow from the continuity of F.

Lemma 1. $\operatorname{sgn}((x-y)(a^2x-b^2y)) = \operatorname{sgn}(\log^2 A - \log^2 B)$.

Proof. Lemma easily follows from the fact that $\operatorname{sgn}(x-y) = \operatorname{sgn}\log\frac{x}{y}$. □

Lemma 2. *For all real t*

$$F_h(-h/2-t)F_h(-h/2+t) = F_h^2(-h/2).$$

Proof.
$F_h^h(-h/2-t)F_h^h(-h/2+t) =$

$$\begin{aligned}
&= y^{2h}\frac{A^{h/2-t}-1}{B^{h/2-t}-1}\cdot\frac{B^{-h/2-t}-1}{A^{-h/2-t}-1}\cdot\frac{A^{h/2+t}-1}{B^{h/2+t}-1}\cdot\frac{B^{-h/2+t}-1}{A^{-h/2+t}-1}\\
&= y^{2h}\frac{B^{-h}}{A^{-h}}\cdot\frac{A^{h/2-t}-1}{B^{h/2-t}-1}\cdot\frac{1-B^{h/2+t}}{1-A^{h/2+t}}\cdot\frac{A^{h/2+t}-1}{B^{h/2+t}-1}\cdot\frac{1-B^{h/2-t}}{1-A^{h/2-t}}\\
&= y^{2h}\left(\frac{x}{y}\right)^h = (xy)^h = F_h^{2h}(-h/2).
\end{aligned}$$

□

Let

$$g(t,A,B) = \frac{A^t\log^2 A}{(A^t-1)^2} - \frac{B^t\log^2 B}{(B^t-1)^2}.$$

Lemma 3.

(1) $g(t,A,B) = g(\pm t, A^{\pm 1}, B^{\pm 1})$,
(2) *g is increasing in t on $(0,\infty)$ if* $\log^2 A - \log^2 B > 0$ *and decreasing otherwise.*

Proof. (1) becomes obvious when we write

$$g(t,A,B) = \frac{\log^2 A}{A^t-2+A^{-t}} - \frac{\log^2 B}{B^t-2+B^{-t}}.$$

From (1) if follows that replacing A, B with A^{-1}, B^{-1} if necessary we may assume that $A, B > 1$. In this case $\operatorname{sgn}(\log^2 A - \log^2 B) = \operatorname{sgn}(A^t - B^t)$.

$$\begin{aligned}\frac{\partial g}{\partial t} &= -\frac{A^t(A^t+1)\log^3 A}{(A^t-1)^3} + \frac{B^t(B^t+1)\log^3 B}{(B^t-1)^3} \\ &= -\frac{1}{t^3}(\phi(A^t) - \phi(B^t)) = -\frac{1}{t^3}(A^t - B^t)\phi'(\xi),\end{aligned}$$

where $\xi > 1$ lies between A^t and B^t and

$$\phi(u) = \frac{u(u+1)\log^3 u}{(u-1)^3}.$$

To complete the proof it is enough to show that $\phi'(u) < 0$ for $u > 1$.

$$\phi'(u) = \frac{(u^2+4u+1)\log^2 u}{(u-1)^4}\left[\frac{3(u^2-1)}{u^2+4u+1} - \log u\right],$$

so the sign of ϕ' is the same as the sign of $\psi(u) = \frac{3(u^2-1)}{u^2+4u+1} - \log u$. But $\psi(1) = 0$ and $\psi'(u) = -(u-1)^4/(u^2+4u+1)^2 < 0$, so $\phi(u) < 0$. □

Let us remind now certain property of convex functions:

Property 4. *If f is convex (concave) then for $h > 0$ the function $g(x) = f(x+h) - f(x)$ is increasing (decreasing). For $h < 0$ the monotonicity of g reverses.*
For log-convex f the same holds for $g(x) = f(x+h)/f(x)$.

4. Proofs

Now we are ready to prove the main results

Proof of Theorem 1. Straightforward computation shows that

$$\begin{aligned}\frac{d^2}{dt^2}\log F_h(t) &= h^{-1}(g(t,A,B) - g(t+h,A,B)) \\ &= h^{-1}(g(|t|,A,B) - g(|t+h|,A,B)) \qquad \text{(by Lemma 3 (1))},\end{aligned}$$

and the assertion follows Lemma 3(2) and from inequality $|t| < |t+h|$ valid if and only if $t > -h/2$ and $h > 0$ or $t < -h/2$ and $h < 0$. □

Proof of Theorem 2. Suppose $r_0 > -h/2$ and F_h is log-convex near r_0 (proof of other cases is similar).

Due to symmetry it is enough to show the inequality

$$F_h(r_0 - t)F_h(r_0 + t) \geq F_n^2(r_0) \tag{8}$$

for $t > 0$.

From theorem 1 we know that F_h is log-convex on $(-h/2, \infty)$, so for t such that $r_0 - t \geq -h/2$ the inequality (8) holds.

If $r_0 - t \leq -h/2$ then $-r_0 + t - h \geq -h/2$ and by lemma 2

$$F_h(r_0 - t)F_h(-r_0 + t - h) = F_h^2(-h/2). \tag{9}$$

From log-convexity we have also

$$F_h(-h/2)F_h(2r_0 + h/2) \geq F_h^2(r_0). \tag{10}$$

Combining (9) and (10) we obtain

$$\begin{aligned} F_h(r_0 - t)F_h(r_0 + t) &= \frac{F_h(r_0 + t)F_h^2(-h/2)}{F_h(-r_0 + t - h)} \\ &\geq \frac{F_h(r_0 + t)F_h(-h/2)F_h^2(r_0)}{F_h(-r_0 + t - h)F_h(2r_0 + h/2)} \\ &\geq F_h^2(r_0), \end{aligned}$$

because

$$\frac{F_h(-h/2)}{F_h(2r_0 + h/2)} \geq \frac{F_h(-r_0 + t - h)}{F_h(r_0 + t)}$$

by property 4. □

5. Examples

In the table below we show some inequalities between classical means:

Harmonic mean	$H = H(x, y) = 2xy/(x + y)$
Geometric mean	$G = G(x, y) = \sqrt{xy}$
Logarithmic mean	$L = L(x, y) = (x - y)/(\log x - \log y)$
Square-mean-root	$Q = Q(x, y) = ((\sqrt{x} + \sqrt{y})/2)^2$
Heronian mean	$N = N(x, y) = (x + \sqrt{xy} + y)/3$
Arithmetic mean	$A = A(x, y) = (x + y)/2$
Centroidal mean	$T = T(x, y) = 2(x^2 + xy + y^2)/3(x + y)$
Root-mean-square	$R = R(x, y) = \sqrt{(x^2 + y^2)/2}$

that can be obtained by appropriate choice of parameters in Theorem 2.

No	Inequality	h	r_0	t	a	b
1	$L^2 \geq GN$	$1/2$	0	1	1	1
2	$L^2 \geq HT$	1	0	2	1	1
3	$Q^2 \geq AG$	$1/2$	0	$1/2$	x	y
4	$Q^2 \geq LN$	$1/2$	$1/2$	$1/2$	1	1
5	$N^2 \geq AL$	1	$1/2$	$1/2$	1	1
6	$A^2 \geq LT$	1	1	1	1	1
7	$A^2 \geq GR$ or $AG \geq HR$	1	0	1	x	y
8	$LN \geq AG$	$1/2$	$1/2$	1	1	1
9	$GN \geq HT$	1	-1	$1/2$	x	y
10	$AN \geq TG$	$1/2$	0	1	x	y
11	$LT \geq HC$	1	1	2	1	1
12	$TA \geq NR$	1	$1/2$	$1/2$	x	y
13	$L^3 \geq AG^2$	1	0	1	1	1
14	$L^3 \geq GQ^2$	$1/2$	$-1/2$	$1/2$	1	1
15	$N^3 \geq AQ^2$	$1/2$	1	$1/2$	1	1
16	$T^3 \geq AR^2$	1	2	1	1	1
17	$LN^2 \geq G^2T$	1	$1/2$	$3/2$	1	1

Note that 4 is stroner than 3 (due to inequality 8), 14 is stronger than 13 (due to 3). Also 1 is stronger than 2 because of 9.

References

[1] Alzer H., *Über eine einparametrige Familie von Mittelwerten*, Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. **1987** (1988), 1-9

[2] Alzer H., *Über Lehmers Mittelwertefamilie*, Elem. Math. **43** (1988), 50-54

[3] Leach E. and Sholander M., *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90.

[4] Stolarsky K. B., *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.

[5] Qi F., *Generalized weighted mean values with two parameters*, Proc. Roy. Soc. London Ser. A **454** (1998), no. 1978, 2723–2732.

[6] Witkowski A., *Monotonicity of generalized extended mean values*, Colloq. Math. 2004 (in print). [ONLINE: RGMIA Research Report Collection, **7**(1), Article 12, 2004 `http:/rgmia.vu.edu.au/v7n1.html`].

[7] Witkowski A., *Weighted extended mean values*, Colloq. Math. 2004 (accepted). [ONLINE: RGMIA Research Report Collection, **7**(1), Article 6, 2004 `http:/rgmia.vu.edu.au/v7n1.html`]

Mielczarskiego 4/29, 85-796 Bydgoszcz, Poland
E-mail address: `alfred.witkowski@atosorigin.com`