The product of divisors minimum and maximum functions

József Sándor
Babeş-Bolyai University of Cluj, Romania

1. Let $T(n) = \prod_{i|n} i$ denote the product of all divisors of n. The product-of-divisors minimum, resp. maximum functions will be defined by

$$\mathcal{T}(n) = \min\{k \geq 1 : \ n|T(k)\} \tag{1}$$

and

$$\mathcal{T}_*(n) = \max\{k \geq 1 : \ T(k)|n\} \tag{2}$$

There are particular cases of the functions F_f^A, G_g^A defined by

$$F_f^A(n) = \min\{k \in A : \ n|f(k)\} \tag{3}$$

and its "dual"

$$G_g^A(n) = \max\{k \in A : \ g(k)|n\}, \tag{4}$$

where $A \subset \mathbb{N}^*$ is a given set, and $f, g : \mathbb{N}^* \to \mathbb{N}$ are given functions, introduced in [8] and [9]. For $A = \mathbb{N}^*$, $f(k) = g(k) = k!$ one obtains the Smarandache function $S(n)$, and its dual $S_*(n)$, given by

$$S(n) = \min\{k \geq 1 : \ n|k!\} \tag{5}$$

and

$$S_*(n) = \max\{k \geq 1 : \ k!|n\} \tag{6}$$

The function $S_*(n)$ has been studied in [8], [9], [4], [1], [3]. For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(k)$, one obtains the Euler minimum, resp. maximum functions

$$E(n) = \min\{k \geq 1 : \ n|\varphi(k)\} \tag{7}$$

studied in [6], [8], [13], resp., its dual

$$E_*(n) = \max\{k \geq 1 : \ \varphi(k)|n\}, \tag{8}$$

studied in [13].

For $A = \mathbb{N}^*$, $f(k) = g(k) = S(k)$ one has the Smarandache minimum and maximum functions

$$S_{min}(n) = \min\{k \geq 1 : \ n|S(k)\}, \tag{9}$$

$$S_{max}(n) = \max\{k \geq 1 : \ S(k)|n\}, \tag{10}$$

introduced, and studied in [15]. The divisor minimum function

$$D(n) = \min\{k \geq 1 : \ n|d(k)\} \tag{11}$$

(where $d(k)$ is the number of divisors of k) appears in [14], while the sum-of-divisors minimum and maximum functions

$$\Sigma(n) = \min\{k \geq 1 : \ n|\sigma(k)\} \tag{12}$$

$$\Sigma_*(n) = \max\{k \geq 1 : \ \sigma(k)|n\} \tag{13}$$

have been recently studied in [16].

For functions $Q(n), Q_1(n)$ obtained from (3) for $f(k) = k!$ and $A =$ set of perfect squares, resp. $A =$ set of squarefree numbers, see [10].

2. The aim of this note is to study some properties of the functions $\mathcal{T}(n)$ and $\mathcal{T}_*(n)$ given by (1) and (2). We note that properties of $T(n)$ in connection with "multiplicatively perfect numbers" have been introduced in [11]. For other asymptotic properties of $T(n)$, see [7]. For divisibility properties of $T(\sigma(n))$ with $T(n)$, see [5]. For asymptotic results of sums of type $\sum_{n \le x} \frac{1}{T(n)}$, see [17].

A divisor i of n is called "unitary" if $\left(i, \frac{n}{i}\right) = 1$. Let $T^*(n)$ be the product of unitary divisors of n. For similar results to [11] for $T^*(n)$, or $T^{**}(n)$ (i.e. the product of "bi-unitary" divisors of n), see [2]. The product of "exponential" divisors $T_e(n)$ is introduced in paper [12]. Clearly, one can introduce functions of type (1) and (2) for $T(n)$ replaced with one of the above functions $T^*(n), T^{**}, T_e(n)$, but these functions will be studied in another paper.

3. The following auxiliary result will be important in what follows.

Lemma 1.

$$T(n) = n^{d(n)/2}, \tag{14}$$

where $d(n)$ is the number of divisors of n.

Proof. This is well-known, see e.g. [11].

Lemma 2.

$$T(a)|T(b) \text{ iff } a|b \tag{15}$$

Proof. If $a|b$, then for any $d|a$ one has $d|b$, so $T(a)|T(b)$. Reciprocally, if $T(a)|T(b)$, let $\gamma_p(a)$ be the exponent of the prime in a. Clearly, if $p|a$, then $p|b$, otherwise $T(a)|T(b)$ is impossible. If $p^{\gamma_p(b)}\|b$, then we must have $\gamma_p(a) \le \gamma_p(b)$. Writing this fact for all prime divisors of a, we get $a|b$.

Theorem 1. *If n is squarefree, then*

$$\mathcal{T}(n) = n \tag{16}$$

Proof. Let $n = p_1 p_2 \dots p_r$, where p_i $(i = \overline{1,r})$ are distinct primes. The relation $p_1 p_2 \dots p_r | T(k)$ gives $p_i | T(k)$, so there is a $d | k$, so that $p_i | d$. But then $p_i | k$ for all $i = \overline{1,r}$, thus $p_1 p_2 \dots p_r = n | k$. Since $p_1 p_2 \dots p_k | T(p_1 p_2 \dots p_k)$, the least k is exactly $p_1 p_2 \dots p_r$, proving (16).

Remark. Thus, if p is a prime, $\mathcal{T}(p) = p$; if $p < q$ are primes, then $\mathcal{T}(pq) = pq$, etc.

Theorem 2. *If $a|b$, $a \neq b$ and b is squarefree, then*

$$\mathcal{T}(ab) = b \tag{17}$$

Proof. If $a|b$, $a \neq b$, then clearly $T(b) = \prod_{d|b} d$ is divisible by ab, so $\mathcal{T}(ab) \leq b$. Reciprocally, if $ab | T(k)$, let $p|b$ a prime divisor of b. Then $p | T(k)$, so (see the proof of Theorem 1) $p|k$. But b being squarefree (i.e. a product of distinct primes), this implies $b|k$. The least such k is clearly $k = b$.

For example, $\mathcal{T}(12) = \mathcal{T}(2 \cdot 6) = 6$, $\mathcal{T}(18) = \mathcal{T}(3 \cdot 6) = 6$, $\mathcal{T}(20) = \mathcal{T}(2 \cdot 10) = 10$.

Theorem 3. *$\mathcal{T}(T(n)) = n$ for all $n \geq 1$.* (18)

Proof. Let $T(n) | T(k)$. Then by (15) one can write $n|k$. The least k with this property is $k = n$, proving relation (18).

Theorem 4. *Let p_i $(i = \overline{1,r})$ be distinct primes, and $\alpha_i \geq 1$ positive integers. Then*

$$\max\left\{\mathcal{T}\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) : i = \overline{1,r}\right\} \leq \mathcal{T}\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq$$
$$\leq l.c.m.[\mathcal{T}(p_1^{\alpha_1}), \dots, \mathcal{T}(p_r^{\alpha_r})] \tag{19}$$

Proof. In [13] it is proved that for $A = \mathbb{N}^*$, and any function f such that $F_f^{\mathbb{N}^*}(n) = F_f(n)$ is well defined, one has

$$\max\{F_f(p_i^{\alpha_i}):\ i = \overline{1,r}\} \leq F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \tag{20}$$

On the other hand, if f satisfies the property

$$a|b \ \Rightarrow\ f(a)|f(b) \quad (a, b \geq 1), \tag{21}$$

then

$$F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq l.c.m.[F_f(p_1^{\alpha_1}), \ldots, F_f(p_r^{\alpha_r})] \tag{22}$$

By Lemma 2, (21) is true for $f(a) = T(a)$, and by using (20), (22), relation (19) follows.

Theorem 5.

$$\mathcal{T}(2^n) = 2^\alpha, \tag{23}$$

where α *is the least positive integer such that*

$$\frac{\alpha(\alpha+1)}{2} \geq n \tag{24}$$

Proof. By (14), $2^n|T(k)$ iff $2^n|k^{d(k)/2}$. Let $k = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, when $d(k) = (\alpha_1+1)\ldots(\alpha_r+1)$. Since $2^{2n}|k^{d(k)} = p_1^{\alpha_1(\alpha_1+1)\ldots(\alpha_r+1)} \ldots p_r^{\alpha_r(\alpha_1+1)\ldots(\alpha_r+1)}$ (let $p_1 < p_2 < \cdots < p_r$), clearly $p_1 = 2$ and the least k is when $\alpha_2 = \cdots = \alpha_r = 0$ and α_1 is the least positive integer with $2n \leq \alpha_1(\alpha_1+1)$. This proves (23), with (24).

For example, $\mathcal{T}(2^2) = 4$, since $\alpha = 2$, $\mathcal{T}(2^3) = 4$ again, $\mathcal{T}(2^4) = 8$ since $\alpha = 3$, etc.

For odd prime powers, the things are more complicated. For example, for 3^n one has:

Theorem 6.

$$\mathcal{T}(3^n) = \min\{3^{\alpha_1}, 2 \cdot 3^{\alpha_2}\}, \tag{25}$$

where α_1 is the least positive integer such that $\dfrac{\alpha_1(\alpha_1+1)}{2} \geq n$, and α_2 is the least positive integer such that $\alpha_2(\alpha_2+1) \geq n$.

Proof. As in the proof of Theorem 5,

$$3^{2n} | p_1^{\alpha_1(\alpha_1+1)\ldots(\alpha_r+1)} \cdot p_2^{\alpha_2(\alpha_1+1)\ldots(\alpha_1+1)} \ldots p_r^{\alpha_r(\alpha_1+1)\ldots(\alpha_r+1)},$$

where $p_1 < p_2 < \cdots < p_r$, so we can distinguish two cases:

a) $p_1 = 2,\ p_2 = 3,\ p_3 \geq 5$

b) $p_1 = 3,\ p_2 \geq 5$.

Then $k = 2^{\alpha_1} \cdot 3^{\alpha_2} \ldots p_r^{\alpha_r} \geq 2^{\alpha_1} \cdot 3^{\alpha_2}$ in case a), and $k \geq 3^{\alpha_1}$ in case b). So for the least k we must have $\alpha_2(\alpha_1+1)(\alpha_2+1) \geq 2n$ with $\alpha_1 = 1$ in case a), and $\alpha_1(\alpha_1+1) \geq 2n$ in case b). Therefore $\dfrac{\alpha_1(\alpha_1+1)}{2} \geq n$ and $\alpha_2(\alpha_2+1) \geq n$, and we must select k with the least of 3^{α_1} and $2^1 \cdot 3^{\alpha_2}$, so Theorem 6 follows.

For example, $\mathcal{T}(3^2) = 6$ since for $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, and $\min\{2 \cdot 3^1, 3^2\} = 6$; $\mathcal{T}(3^3) = 9$ since for $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 2$ and $\min\{2 \cdot 3^2, 3^2\} = 9$.

Theorem 7. *Let $f : [1, \infty) \to [0, \infty)$ be given by $f(x) = \sqrt{x} \log x$. Then*

$$f^{-1}(\log n) < \mathcal{T}(n) \leq n \tag{26}$$

for all $n \geq 1$, where f^{-1} denotes the inverse function of f.

Proof. Since $n | T(n)$, the right side of (26) follows by definition (1) of $\mathcal{T}(n)$. On the other hand, by the known inequality $d(k) < 2\sqrt{k}$, and Lemma 1 (see (14)) we get $T(k) < k^{\sqrt{k}}$, so $\log T(k) < \sqrt{k} \log k = f(k)$. Since $n | T(k)$ implies $n \leq T(k)$, so $\log n \leq \log T(k) < f(k)$, and the function f being strictly increasing and continuous, by the bijectivity of f, the left side of (26) follows.

4. The function $\mathcal{T}_*(n)$ given by (2) differs in many aspects from $\mathcal{T}(n)$. The first such property is:

Theorem 8. *$\mathcal{T}_*(n) \leq n$ for all n, with equality only if $n = 1$ or $n =$ prime.*

Proof. If $T(k)|n$, then $T(k) \leq n$. But $T(k) \geq k$, so $k \leq n$, and the inequality follows.

Let us now suppose that for $n > 1$, $\mathcal{T}_*(n) = n$. Then $T(n)|n$, by definition 2. On the other hand, clearly $n|T(n)$, so $T(n) = n$. This is possible only when $n =$ prime.

Remark. Therefore the equality

$$\mathcal{T}_*(n) = n \quad (n > 1)$$

is a characterization of the prime numbers.

Lemma 3. *Let $p_1, \ldots, p_r$ be given distinct primes ($r \geq 1$). Then the equation*

$$T(k) = p_1 p_2 \ldots p_r$$

is solvable iff $r = 1$.

Proof. Since $p_i|T(k)$, we get $p_i|k$ for all $i = \overline{1, r}$. Thus $p_1 \ldots p_r|k$, and Lemma 2 implies $T(p_1 \ldots p_r)|T(k) = p_1 \ldots p_r$. Since $p_1 \ldots p_r|T(p_1 \ldots p_r)$, we have $T(p_1 \ldots p_r) = p_1 \ldots p_r$, which by Theorem 8 is possible only if $r = 1$.

Theorem 9. *Let $P(n)$ denote the greatest prime factor of $n > 1$. If n is squarefree, then*

$$\mathcal{T}_*(n) = P(n) \tag{27}$$

Proof. Let $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$. If $T(k) | (p_1 \dots p_r)$, then clearly $T(k) \in \{1, p_1, \dots, p_r, p_1 p_2, \dots, p_1 p_2 \dots p_r\}$. By Lemma 3 we cannot have $T(k) \in \{p_1 p_2, \dots, p_1 p_2 \dots p_r\}$, so $T(k) \in \{1, p_1, \dots, p_r\}$, when $k \in \{1, p_1, \dots, p_r\}$. The greatest k is $p_r = P(n)$.

Remark. Therefore $\mathcal{T}_*(pq) = q$ for $p < q$. For example, $\mathcal{T}_*(2 \cdot 7) = 7$, $\mathcal{T}_*(3 \cdot 5) = 5$, $\mathcal{T}_*(3 \cdot 7) = 7$, $\mathcal{T}_*(2 \cdot 11) = 11$, etc.

Theorem 10.

$$\mathcal{T}_*(p^n) = p^\alpha \quad (p = prime) \tag{28}$$

where α is the greatest integer with the property

$$\frac{\alpha(\alpha+1)}{2} \le n \tag{29}$$

Proof. If $T(k) | p^n$, then $T(k) = p^m$ for $m \le n$. Let q be a prime divisor of k. Then $q = T(q) | T(k) = 2^m$ implies $q = p$, so $k = p^\alpha$. But then $T(k) = p^{\alpha(\alpha+1)/2}$ with α the greatest number such that $\alpha(\alpha+1)/2 \le n$, which finishes the proof of (28).

For example, $\mathcal{T}_*(4) = 2$, since $\dfrac{\alpha(\alpha+1)}{2} \le 2$ gives $\alpha_{max} = 1$.

$\mathcal{T}_*(16) = 4$, since $\dfrac{\alpha(\alpha+1)}{2} \le 4$ is satisfied with $\alpha_{max} = 2$.

$\mathcal{T}_*(9) = 3$, and $\mathcal{T}_*(27) = 9$ since $\dfrac{\alpha(\alpha+1)}{2} \le 3$ with $\alpha_{max} = 2$.

Theorem 11. *Let p, q be distinct primes. Then*

$$\mathcal{T}_*(p^2 q) = \max\{p, q\} \tag{30}$$

Proof. If $T(k) | p^2 q$, then $T(k) \in \{1, p, q, p^2, pq, p^2 q\}$. The equations $T(k) = p^2$, $T(k) = pq$, $T(k) = p^2 q$ are impossible. For example, for the first equation, this can be proved as follows. By $p | T(k)$ one has $p | k$, so $k = pm$. Then $p(pm)$ are in $T(k)$, so $m = 1$. But then $T(k) = p \ne p^2$.

For the last equation, $k = (pq)m$ and $pqm(pm)(qm)(pqm)$ are in $T(k)$, which is impossible.

Theorem 12. *Let p, q be distinct primes. Then*

$$\mathcal{T}_*(p^3 q) = \max\{p^2, q\} \tag{31}$$

Proof. As above, $T(k) \in \{1, p, q, pq, p^2 q, p^3 q, p^2, p^3\}$ and $T(k) \in \{pq, p^2 q, p^3 q, p^2\}$ are impossible. But $T(k) = p^3$ by Lemma 1 gives $k^{d(k)} = p^6$, so $k = p^m$, when $d(k) = m + 1$. This gives $m(m + 1) = 6$, so $m = 2$. Thus $k = p^2$. Since $p < p^2$ the result follows.

Remark. The equation

$$T(k) = p^s \tag{32}$$

can be solved only if $k^{d(k)} = p^{2s}$, so $k = p^m$ and we get $m(m + 1) = 2s$. Therefore $k = p^m$, with $m(m + 1) = 2s$, if this is solvable. If s is not a **triangular number**, this is impossible.

Theorem 13. *Let p, q be distinct primes. Then*

$$\mathcal{T}_*(p^s q) = \begin{cases} \max\{p, q\}, & \text{if } s \text{ is not a triangular number}, \\ \max\{p^n, q\}, & \text{if } s = \dfrac{m(m+1)}{2}. \end{cases}$$

References

[1] K. Atanasov, *Remark on József Sándor and Florian Luca's theorem*, C. R. Acad. Bulg. Sci. **55**(2002), no. 10, 9-14.

[2] A. Bege, *On multiplicatively unitary perfect numbers*, Seminar on Fixed Point Theory, Cluj-Napoca, **2**(2001), 59-63.

[3] M. Le, *A conjecture concerning the Smarandache dual function*, Smarandache Notion J. **14**(2004), 153-155.

[4] F. Luca, *On a divisibility property involving factorials*, C. R. Acad. Bulg. Sci. **53**(2000), no. 6, 35-38.

[5] F. Luca, *On the product of divisors of n and $\sigma(n)$*, J. Ineq. Pure Appl. Math. **4**(2003), no. 2, 7 pp. (electronic).

[6] P. Moree and H. Roskam, *On an arithmetical function related to Euler's totient and the discriminantor*, Fib. Quart. **33**(1995), 332-340.

[7] T. Šalát and J. Tomanová, *On the product of divisors of a positive integer*, Math. Slovaca **52**(2002), no. 3, 271-287.

[8] J. Sándor, *On certain generalizations of the Smarandache function*, Notes Number Theory Discr. Math. **5**(1999), no. 2, 41-51.

[9] J. Sándor, *On certain generalizations of the Smarandache function*, Smarandache Notions J. **11**(2000), no. 1-3, 202-212.

[10] J. Sándor, *The Smarandache function of a set*, Octogon Math. Mag. **9**(2001), no. 1B, 369-371.

[11] J. Sándor, *On multiplicatively perfect numbers*, J. Ineq. Pure Appl. Math. **2**(2001), no. 1, 6 pp. (electronic).

[12] J. Sándor, *On multiplicatively e-perfect numbers*, (to appear).

[13] J. Sándor, *On the Euler minimum and maximum functions*, (to appear).

[14] J. Sándor, *A note on the divisor minimum function*, (to appear).

[15] J. Sándor, *The Smarandache minimum and maximum functions*, (to appear).

[16] J. Sándor, *The sum-of-divisors minimum and maximum functions*, (to appear).

[17] Z. Weiyi, *On the divisor product sequence*, Smarandache Notions J., **14**(2004), 144-146.