On the Certain Extensions of Hilbert Inequality

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Abstract In this paper we establish some new inequalities similar to certain extensions of Hilbert inequality.

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1 Introduction

In[1,P.284] the following the extension of Hilbert inequality is given

Theorem A *If* $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} \geq 1, 0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} \leq 1$, *then*

$$\sum_1^\infty \sum_1^\infty \frac{a_m b_n}{(m+n)^\lambda} \leq k\Big(\sum_1^\infty a_m^p\Big)^{1/p}\Big(\sum_1^\infty b_n^q\Big)^{1/q}, \tag{1}$$

where $k = k(p,q)$ *depends on* p *and* q *only.*

The integral analogue of Theorem A can be stated as follows[1,P.286]

Theorem B *Under the same conditions as in Theorem A ,we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leq k\Big(\int_0^\infty p^p(x)dx\Big)^{1/p}\Big(\int_0^\infty g^q(y)dy\Big)^{1/q}, \tag{2}$$

where $k = k(p,q)$ *depends on* p *and* q *only.*

The inequalities in Theorems A and B were studied extensively and numerous variants, generalizations, and extensions appeared in the literature,see[2-7]. Recently, in[8] inqualities have given similar to the inequalities given in Theorems A and B. The main purpose of this paper is to establish some new inequalities similar to Theorems A and B, too. Our results provide new estimates on inequalities of this type.

2 Main Results

Our main results are given in the following Theorems

Theorem 1 *Let* $a(s)$ *and* $b(t)$ *be real-valued nonnegative non-decreasing functions defined on* N_m *and* N_n*,respectively, where* $N_m = \{0,1,2,\cdots m\}, N_n = \{0,1,2,\cdots,n\}$ *and define the operator* ∇ by

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$\nabla u(t) = u(t) - u(t-1)$ for any non-decreasing function u defined on $N_0 = \{0, 1, 2, \cdots\}$. Let $p \geq 1, q \geq 1$ and $h > 1, \frac{1}{h} + \frac{1}{l} = 1$. Then

$$\sum_{s=1}^{m}\sum_{t=1}^{n} \frac{hl\Big(a(s)-a(0)\Big)^p \Big(b(t)-b(0)\Big)^q}{l\cdot s^{h-1} + h\cdot t^{l-1}}$$

$$\leq pq \cdot m^{(h-1)/h} \cdot n^{(l-1)/l} \left(\sum_{s=1}^{m} (m-s+1)\left(\nabla a(s)\cdot \Big(a(s)-a(0)\Big)^{p-1}\right)^h \right)^{1/h}$$

$$\times \left(\sum_{t=1}^{n} (n-t+1)\left(\nabla b(t)\cdot \Big(b(t)-b(0)\Big)^{q-1}\right)^l \right)^{1/l}. \tag{3}$$

Proof: From the hypotheses, it is easy to observe that

$$a(s) - a(0) = \sum_{\tau=1}^{s} \nabla a(\tau), s \in N_m \tag{4}$$

$$b(t) - b(0) = \sum_{\sigma=1}^{t} \nabla b(\sigma), t \in N_n \tag{5}$$

By using the elementary inequality[1,P.40], $x^p - y^q \leq px^{p-1}(x-y)$, where $x \geq 0, y \geq 0$ and $p \geq 1$, we have

$$\Big(a(s+1)-a(0)\Big)^p - \Big(a(s)-a(0)\Big)^p \leq p\Big(a(s+1)-a(0)\Big)^{p-1}\Big(a(s+1)-a(s)\Big)$$

$$= p\Big(a(s+1)-a(0)\Big)^{p-1} \cdot \nabla a(s+1)$$

and

$$\sum_{s=0}^{k-1}\left(\Big(a(s+1)-a(0)\Big)^p - \Big(a(s)-a(0)\Big)^p\right) = \Big(a(k)-a(0)\Big)^p$$

$$\leq p\sum_{s=0}^{k-1} \nabla a(s+1)\cdot\Big(a(s+1)-a(0)\Big)^{p-1} = p\sum_{s=1}^{k}\nabla a(s)\cdot\Big(a(s)-a(0)\Big)^{p-1}$$

Thus

$$\Big(a(s)-a(0)\Big)^p \leq p\sum_{\tau=1}^{s}\nabla a(\tau)\cdot\Big(a(\tau)-a(0)\Big)^{p-1} \tag{6}$$

and similarly

$$\Big(b(t)-b(0)\Big)^q \leq q\sum_{\sigma=1}^{t}\nabla b(\sigma)\cdot\Big(b(\sigma)-b(0)\Big)^{q-1} \tag{7}$$

From (6),(7) and using Hölder inequality and the elementary inequality[9]

$$xy \leq \frac{x^h}{h} + \frac{y^l}{l} \tag{8}$$

where $x \geq 0, y \geq 0$ and $\frac{1}{h} + \frac{1}{l} = 1, h > 1$, then

$$\Big(a(s)-a(0)\Big)^p\Big(b(t)-b(0)\Big)^q \leq pq\sum_{\tau=1}^{s}\nabla a(\tau)\cdot\Big(a(\tau)-a(0)\Big)^{p-1}$$

$$\times \sum_{\sigma=1}^{t} \nabla b(\sigma) \cdot \Big(b(\sigma) - b(0)\Big)^{q-1}$$

$$\leq pq \cdot s^{(h-1)/h} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot \Big(a(\tau) - a(0)\Big)^{p-1} \right)^{h} \right)^{1/h}$$

$$\times t^{(l-1)/l} \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot \Big(b(\sigma) - b(0)\Big)^{q-1} \right)^{l} \right)^{1/l}$$

$$\leq \frac{pq(l \cdot s^{h-1} + h \cdot t^{l-1})}{hl} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot \Big(a(\tau) - a(0)\Big)^{p-1} \right)^{h} \right)^{1/h}$$

$$\times \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot \Big(b(\sigma) - b(0)\Big)^{q-1} \right)^{l} \right)^{1/l} \tag{9}$$

Dividing both sides of (9) by $\frac{l \cdot s^{h-1} + h \cdot t^{l-1}}{hl}$ and then taking the sum over t from 1 to n and then the sum over s from 1 to m and using Hölder inequality ,we observe that

$$\sum_{s=1}^{m} \sum_{t=1}^{n} \frac{hl\Big(a(s) - a(0)\Big)^{p} \Big(b(t) - b(0)\Big)^{q}}{l \cdot s^{h-1} + h \cdot t^{l-1}} \leq pq \sum_{s=1}^{m} \left(\sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot \Big(a(\tau) - a(0)\Big)^{p-1} \right)^{h} \right)^{1/h}$$

$$\times \sum_{t=1}^{n} \left(\sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot \Big(b(\sigma) - b(0)\Big)^{q-1} \right)^{l} \right)^{1/l}$$

$$\leq pq \cdot m^{(h-1)/h} \left(\sum_{s=1}^{m} \sum_{\tau=1}^{s} \left(\nabla a(\tau) \cdot \Big(a(\tau) - a(0)\Big)^{p-1} \right)^{h} \right)^{1/h}$$

$$\times n^{(l-1)/l} \left(\sum_{t=1}^{n} \sum_{\sigma=1}^{t} \left(\nabla b(\sigma) \cdot \Big(b(\sigma) - b(0)\Big)^{q-1} \right)^{l} \right)^{1/l}$$

$$= pq \cdot m^{(h-1)/h} \cdot n^{(l-1)/l} \left(\sum_{\tau=1}^{m} \left(\nabla a(\tau) \cdot \Big(a(\tau) - a(0)\Big)^{p-1} \right)^{h} \sum_{s=\tau}^{m} 1 \right)^{h-1}$$

$$\times \left(\sum_{\sigma=1}^{n} \left(\nabla b(\sigma) \cdot \Big(b(\sigma) - b(0)\Big)^{q-1} \right)^{l} \sum_{t=\sigma}^{n} 1 \right)^{l-1}$$

$$= pq \cdot m^{(h-1)/h} \cdot n^{(l-1)/l} \left(\sum_{s=1}^{m} (m - s + 1) \left(\nabla a(s) \cdot \Big(a(s) - a(0)\Big)^{p-1} \right)^{h} \right)^{h-1}$$

$$\times \left(\sum_{t=1}^{n} (n - t + 1) \left(\nabla b(t) \cdot \Big(b(t) - b(0)\Big)^{q-1} \right)^{l} \right)^{l-1}$$

Remark 1: We take $p = q = 1, a(0) = b(0) = 0$ in (3), the inequality (3) reduces to the following inequality

$$\sum_{s=1}^{m} \sum_{t=1}^{n} \frac{a(s)b(t)}{l \cdot s^{h-1} + h \cdot t^{l-1}} \leq \frac{m^{(h-1)/h} \cdot n^{(l-1)/l}}{hl} \left(\sum_{s=1}^{m} (m - s + 1) \Big(\nabla a(s)\Big)^{h} \right)^{1/h}$$

$$\times\left(\sum_{t=1}^{n}(n-t+1)\Big(\nabla b(t)\Big)^{l}\right)^{1/l} \tag{10}$$

This is just a new inequality similar to Theorem 1 which was given by B.G.Pachpatte in [8].

On the other hand, dividing both sides of (3) by $m^{(h-1)/h}\cdot n^{(l-1)/l}$ and then taking the sum over n from 1 to v and then the sum over m from 1 to u and using Hölder inequality ,we get following inequality

$$\sum_{m=1}^{u}\sum_{n=1}^{v}\left(\frac{m^{(1-h)/h}}{n^{(l-1)/l}}\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{hl\Big(a(s)-a(0)\Big)^{p}\Big(b(t)-b(0)\Big)^{q}}{l\cdot s^{h-1}+h\cdot t^{l-1}}\right)$$

$$\leq pq\cdot u^{(h-1)/h}\cdot v^{(l-1)/l}\left(\sum_{s=1}^{u}(u-s+1)(m-s+1)\bigg(\nabla a(s)\cdot\Big(a(s)-a(0)\Big)^{p-1}\bigg)^{h}\right)^{1/h}$$

$$\times\left(\sum_{t=1}^{v}(v-t+1)(n-t+1)\bigg(\nabla b(t)\cdot\Big(b(t)-b(0)\Big)^{q-1}\bigg)^{l}\right)^{1/l}. \tag{11}$$

where u,v are two nature numbers.

Theorem 2 *Let $f(s)$ and $g(t)$ be two real-valued nonnegative, non-decreasing continuous functions defined on $[0,x)$ and $[0,y)$, respectively, where x and y are positive real numbers. Let $p\geq 1, q\geq 1$ and $\frac{1}{h}+\frac{1}{l}=1, h>1$, then*

$$\int_0^x\int_0^y\frac{hl\Big(f^p(s)-f^p(0)\Big)\Big(g^q(t)-g^q(0)\Big)}{l\cdot s^{h-1}+h\cdot t^{l-1}}dsdt\leq pqx^{(h-1)/h}\cdot y^{(l-1)/l}$$

$$\times\left(\int_0^x(x-s)\Big(f'(s)f^{p-1}(s)\Big)^{h}ds\right)^{1/h}\left(\int_0^y(y-t)\Big(g'(t)g^{q-1}(t)\Big)^{l}dt\right)^{1/l}. \tag{12}$$

Proof: From the hypotheses ,we have

$$f^p(s)-f^p(0)=p\int_0^s f'(\tau)f^{p-1}(\tau)d\tau, s\in[0,x), \tag{13}$$

$$g^q(t)-g^q(0)=q\int_0^t g'(\sigma)g^{q-1}(\sigma)d\sigma, t\in[0,y). \tag{14}$$

From (13) and (14) and using Hölder integral inequality and the elementary inequality (8), we have

$$\Big(f^p(s)-f^p(0)\Big)\Big(g^q(t)-g^q(0)\Big)\leq pq\cdot s^{(h-1)/h}\left(\int_0^s\Big(f'(\tau)f^{p-1}(\tau)\Big)^{h}d\tau\right)^{1/h}$$

$$\times t^{(l-1)/l}\left(\int_0^t\Big(g'(\sigma)g^{q-1}(\sigma)\Big)^{l}d\sigma\right)^{1/l}$$

$$\leq pq\frac{l\cdot s^{h-1}+h\cdot t^{l-1}}{hl}\left(\int_0^s\Big(f'(\tau)f^{p-1}(\tau)\Big)^{h}d\tau\right)^{1/h}$$

$$\times\left(\int_0^t\Big(g'(\sigma)g^{q-1}(\sigma)\Big)^{l}d\sigma\right)^{1/l} \tag{15}$$

Dividing both sides of (15) by $\frac{l \cdot s^{h-1} + h \cdot t^{l-1}}{hl}$ and integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Hölder integral inequality,we get that

$$\int_0^x \int_0^y \frac{hl\Big(f^p(s) - f^p(0)\Big)\Big(g^q(t) - g^q(0)\Big)}{l \cdot s^{h-1} + h \cdot t^{l-1}} dsdt$$

$$\leq pq \int_0^x \left(\int_0^s \Big(f'(\tau) f^{p-1}(\tau)\Big)^h d\tau \right)^{1/h} ds \times \int_0^y \left(\int_0^t \Big(g'(\sigma) g^{q-1}(\sigma)\Big)^l d\sigma \right)^{1/l} dt$$

$$\leq pqx^{(h-1)/h} \cdot y^{(l-1)/l} \left(\int_0^x \left(\int_0^s \Big(f'(\tau) f^{p-1}(\tau)\Big)^h d\tau \right) ds \right)^{1/h} \left(\int_0^y \left(\int_0^t \Big(g'(\sigma) g^{q-1}(\sigma)\Big)^l d\sigma \right) dt \right)^{1/l}$$

$$= pq \cdot x^{(h-1)/h} \cdot y^{(l-1)/l} \left(\int_0^x (x-s) \Big(f'(s) f^{p-1}(s)\Big)^h ds \right)^{1/h} \left(\int_0^y (y-t) \Big(g'(t) g^{q-1}(t)\Big)^l dt \right)^{1/l}$$

Remark 2 If we take $p = q = 1, f(0) = g(0) = 0$ in (12), then

$$\int_0^x \int_0^y \frac{f(s)g(t)}{l \cdot s^{h-1} + h \cdot l^{l-1}} dsdt \leq \frac{x^{(h-1)/h} \cdot y^{(l-1)/l}}{hl}$$

$$\times \left(\int_0^x (x-s) \Big(f'(s)\Big)^h ds \right)^{1/h} \left(\int_0^y (y-t) \Big(g'(t)\Big)^l dt \right)^{1/l}. \qquad (16)$$

This is just a new inequality similar to Theorem 2 which was given by B.G.Pachpatte in [8].

On the other hand,we apply the inequality (8) on the right-hand side of (12), we get that

$$\int_0^x \int_0^y \frac{hl\Big(f^p(s) - f^p(0)\Big)\Big(g^q(t) - g^q(0)\Big)}{l \cdot s^{h-1} + h \cdot t^{l-1}} dsdt \leq pqx^{(h-1)/h} \cdot y^{(l-1)/l}$$

$$\times \left(\frac{1}{h} \int_0^x (x-s) \Big(f'(s) f^{p-1}(s)\Big)^h ds + \frac{1}{l} \int_0^y (y-t) \Big(g'(t) g^{q-1}(t)\Big)^l dt \right). \qquad (17)$$

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