On Refinements of Reverse Hilbert Type Integral Inequalities

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Abstract Some reverse Hilbert's type inequalities are improved.

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1 Introduction

In recent years several authors [1], [2], [4], [5], [6], [7], [8], [9] and [10] have gaven considerable attention to Hilbert's inequalities and Hilbert's type inequalities and their various generalizations. In particular, Zhao and Bencze[11] established the inverses of two new inequalities similar to Hilbert's inequality[10,P.226]. The main purpose of this paper is to improve these two reverse inequalities.

2 Main results

Our main results are given in the following theorems.

Theorem 1 *Let* $h_i \geq 1$ *and* $f_i(\sigma_i) > 0$ for $\sigma_i \in (0, x_i)$ *where* x_i *are positive real numbers and define* $F_i(s_i) = \int_0^{s_i} f_i(\sigma_i)d\sigma_i$ *and* for $s_i \in (0, x_i)$ *and* $\frac{1}{p_i} + \frac{1}{q_i} = 1, p_i < 0$ or $0 < p_i < 1$, *where* $i = 1, \ldots, n$. *Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{C(s_i, t_i, p_i)} ds_1 \cdots ds_n \geq \prod_{i=1}^n h_i x_i^{1/p_i} \left(\int_0^{x_i} \Big(x_i - s_i \Big)^{q_i} ds_i \right)^{1/q_i}, \quad (1)$$

where $C(s_i, p_i) = \prod_{i=1}^n (\int_0^{s_i} f_i^{p_i}(\sigma_i) d\sigma_i)^{1/p_i}$.

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Proof From the hypotheses,it is easy to observe that

$$F_i^{h_i}(s_i)=h_i\int_0^{s_i}F_i^{h_i-1}(\sigma_i)f_i(\sigma_i)d\sigma_i, s_i\in(0,x_i).$$

Therefore

$$\prod_i^n F_i^{h_i}(s_i)=\prod_i^n h_i\Big(\int_0^{s_i}F_i^{h_i-1}(\sigma_i)f_i(\sigma_i)d\sigma_i\Big) \tag{2}$$

On the other hand,according to Holder integral inequality(see[2] P.154) we have

$$\int_0^{s_i}F_i^{h_i-1}(\sigma_i)f_i(\sigma_i)d\sigma_i\geq\Big(\int_0^{s_i}\Big(F_i^{h_i-1}(\sigma_i)\Big)^{q_i}d\sigma_i\Big)^{1/q_i}\cdot\Big(\int_0^{s_i}f_i^{p_i}(\sigma_i)d\sigma_i\Big)^{1/p_i} \tag{3}$$

By (2) and (3) yield that

$$\prod_{i=1}^n F_i^{h_i}(s_i)\geq\prod_{i=1}^n h_i\Big(\int_0^{s_i}f_i^{p_i}(\sigma_i)d\sigma_i\Big)^{1/p_i}\Big(\int_0^{s_i}\Big(F_i^{h_i-1}(\sigma)\Big)^{q_i}d\sigma_i\Big)^{1/q_i}.$$

Thus

$$\frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{C(s_i,p_i)}ds_1\cdots ds_n\geq\prod_{i=1}^n h_i\Big(\int_0^{s_i}\Big(F_i^{h_i-1}(\sigma_i)\Big)^{q_i}d\sigma_i\Big)^{1/q_i}, \tag{4}$$

where $C(s_i,p_i)=\prod_{i=1}^n(\int_0^{s_i}f_i^{p_i}(\sigma_i)d\sigma_i)^{1/p_i}$.

Integrating both sides of (4) over s_i from 0 to $x_i(i=1,\ldots,n)$ and using special case of Holder integral inequality, we observe that

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{C(s_i,p_i)}ds_1\cdots ds_n\geq\prod_{i=1}^n h_i\Big(\int_0^{x_i}\Big(\int_0^{s_i}\Big(F_i^{h_i-1}(\sigma_i)\Big)^{q_i}d\sigma_i\Big)^{1/q_i}ds_i\Big)$$

$$\geq\prod_{i=1}^n h_ix_i^{1/p_i}\Big(\int_0^{x_i}\Big(\int_0^{s_i}\Big(F_i^{h_i-1}(\sigma_i)\Big)^{q_i}d\sigma_i\Big)ds_i\Big)^{1/q_i}$$

$$=\prod_{i=1}^n h_ix_i^{1/p_i}\Big(\int_0^{x_i}\Big(x_i-s_i\Big)\Big(F_i^{h_i-1}(s_i)\Big)^{q_i}ds_i\Big)^{1/q_i}$$

The proof is complete.

Remark 1 Taking $n=2,x_1=x,y=x_2,s_1=s,s_2=t,h_1=h,h_2=l,p_{1,2}=p,q_{1,2}=q,F_1(s_1)=F(s)$ and $F_2(s_2)=G(t)$ to (1), (1) changes to the following.

$$\int_0^x\int_0^y\frac{F^h(s)G^l(t)}{C(s,t,p)}dsdt\geq hl(xy)^{1/p}\Big(\int_0^x\Big(x-s\Big)\Big(F^{h-1}(s)\Big)^q ds\Big)^{1/q}$$

$$\times\Big(\int_0^y\Big(y-t\Big)\Big(G^{l-1}(t)\Big)^q dt\Big)^{1/q},$$

where $C(s,t,p)=(\int_0^s f^p(\sigma)d\sigma)^{1/p}(\int_0^t g^p(\tau)d\tau)^{1/p}$.

This is just a new inequality which was given by Zhao and Bencze[11].

Theorem 2 *Let* f_i,F_i *be as in Theorem 1. Let* $p_i(\sigma_i)$ *be n positive functions defined for* $\sigma_i\in(0,x_i)$ *and define* $P_i(s_i)=\int_0^{s_i} p_i(\sigma_i)d\sigma_i$, for $s_i\in(0,x_i)$, *where* x_i *are n positive real numbers and* p_i,q_i *are n real numbers and* $\frac{1}{p_i}+\frac{1}{q_i}=1, p_i<0$ or $0<p_i<1$. *Let* ϕ_i *be n real-valued nonnegative, concave, and supermultiplicative functions* (f_i *is said to be supermultiplicative function if* $f(x_1x_2)\geq f(x_1)f(x_2), x_1,x_2\in R_+$) *defined on* $R_+=[0,+\infty)$ *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n\phi_i\Big(F_i(s_i)\Big)}{D(s_i,p_i)}ds_1\cdots ds_n$$

$$\geq L(x_i,p_i)\prod_{i=1}^n\left(\int_0^{x_i}\Big(x_i-s_i\Big)\Big(\phi_i\Big(\frac{f_i(s_i)}{p_i(s_i)}\Big)\Big)^{q_i}ds_i\right)^{1/q_i}, \tag{5}$$

where

$$L(x_i,p_i)=\prod_{i=1}^n\left(\int_0^{x_i}\Big(\frac{\phi_i\Big(P_i(s_i)\Big)}{P_i(s_i)}\Big)^{p_i}ds_i\right)^{1/p_i}$$

and

$$D(s_i,p_i)=\prod_{i=1}^n\left(\int_0^{s_i}p_i^{p_i}(\sigma_i)d\sigma_i\right)^{1/p_i}.$$

Proof From the hypotheses and by using Jensen inquality and Holder integral inequality , it is easy to observe that

$$\phi_i\left(F_i(s_i)\right)=\phi_i\left(\frac{P_i(s_i)\int_0^{s_i}p_i(\sigma_i)\frac{f_i(\sigma_i)}{p_i(\sigma_i)}d\sigma_i}{\int_0^{s_i}p_i(\sigma_i)d\sigma_i}\right)$$

$$\geq\phi_i(P_i(s_i))\phi_i\left(\frac{\int_0^{s_i}p_i(\sigma_i)\frac{f_i(\sigma_i)}{p_i(\sigma_i)}d\sigma_i}{\int_0^{s_i}p_i(\sigma_i)d\sigma_i}\right)$$

$$\geq\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\int_0^{s_i}p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)d\sigma_i$$

$$\geq\left(\frac{\phi_i\Big(P_i(s_i)\Big)}{P_i(s_i)}\right)\left(\int_0^{s_i}p_i^{p_i}(\sigma_i)d\sigma_i\right)^{1/p_i}\left(\int_0^{s_i}\Big(\phi_i\Big(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\Big)\Big)^{q_i}d\sigma_i\right)^{1/q_i}. \tag{6}$$

Hence,we get that

$$\frac{\prod_{i=1}^{n}\phi_i\Big(F_i(s_i)\Big)}{D(s_i,p_i)} \geq \prod_{i=1}^{n}\left(\frac{\phi_i\Big(P_i(s_i)\Big)}{P_i(s_i)}\right)\left(\int_0^{s_i}\Big(\phi_i\Big(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\Big)\Big)^{q_i}d\sigma_i\right)^{1/q_i}, \tag{7}$$

where $D(s_i,p_i)=\prod_{i=1}^{n}\left(\int_0^{s_i}p_i^{p_i}(\sigma_i)d\sigma_i\right)^{1/p_i}$.

Integrating two sides of (7) over s_i from 0 to x_i and using Holder integral inequality, we observe that

$$\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}\phi_i\Big(F_i(s_i)\Big)}{D(s_i,p_i)}ds_1\cdots ds_n\\
&\geq \prod_{i=1}^{n}\left(\int_0^{x_i}\frac{\phi_i\Big(P_i(s_i)\Big)}{P_i(s_i)}\Big(\int_0^{s_i}\Big(\phi_i\Big(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\Big)\Big)^{q_i}d\sigma_i\Big)^{1/q_i}ds_i\right)\\
&\geq \prod_{i=1}^{n}\left(\int_0^{x_i}\Big(\frac{\phi_i\Big(P_i(s_i)\Big)}{P_i(s_i)}\Big)^{p_i}ds_i\right)^{1/p_i}\left(\int_0^{x_i}\Big(\int_0^{s_i}\Big(\phi_i\Big(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\Big)\Big)^{q_i}d\sigma_i\Big)ds_i\right)^{1/q_i}\\
&= L(x_i,p_i)\prod_{i=1}^{n}\left(\int_0^{x_i}\Big(x_i-s_i\Big)\Big(\phi_i\Big(\frac{f_i(s_i)}{p(s_i)}\Big)\Big)^{q_i}ds_i\right)^{1/q_i},
\end{aligned}$$

where

$$L(x_i,p_i)=\prod_{i=1}^{n}\left(\int_0^{x_i}\Big(\frac{\phi_i\Big(P_i(s_i)\Big)}{P_i(s_i)}\Big)^{p_i}ds_i\right)^{1/p_i}.$$

Remark 2 Taking $n=2, x_1=x, y=x_2, s=s_1, t=s_2, h_1=h, h_2=l, p_{1,2}=p, q_{1,2}=q, \phi_1(F_1(s_1))=\phi(F(s)), \phi_2(F_2(s_2))=\psi(G(t)), F_1(s_1)=F(s)$ and $F_2(s_2)=G(t)$ to (5), (5) changes to the following.

$$\begin{aligned}
\int_0^{x}\int_0^{y}\frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{D(s,t,p)}dsdt &\geq L(x,y,p)\left(\int_0^{x}\Big(x-s\Big)\Big(\phi\Big(\frac{f(s)}{p(s)}\Big)\Big)^{q}ds\right)^{1/q}\\
&\times\left(\int_0^{y}\Big(y-t\Big)\Big(\psi\Big(\frac{g(t)}{q(t)}\Big)\Big)^{q}dt\right)^{1/q},
\end{aligned}$$

where

$$L(x,y,p)=\left(\int_0^{x}\Big(\frac{\phi\Big(P(s)\Big)}{P(s)}\Big)^{p}ds\right)^{1/p}\left(\int_0^{y}\Big(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\Big)^{p}dt\right)^{1/p}$$

and

$$D(s,t,p)=\left(\int_0^s p^p(\sigma)d\sigma\right)^{1/p}\left(\int_0^t q^p(\tau)d\tau\right)^{1/p}$$

This is just a another new inequality which was given by Zhao and Bencze[11].

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References

[1] B. G. Pachpatte, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.,* 1998, **226**, 166-179.

[2] G. H. Hardy, J. E. Littlewood and P*ó*lya, Inequalities, Cambridge Univ. Press, Cambridge, U. K. 1934.

[3] D.S.Mitrinovic. Analytic inequalities, Springer-verlag,1970.

[4] Gao Minzhe. On Hilbert inequality and its Applications. J.Math.Anal.Appl. 1997**212**,316-323.

[5] HuKe. On Hilbert inequality and its application, Advances Mathernatics.1993,**22**,160-163.

[6] Yang Bicheng and L.Debnath. Generalizations of Hardy integral inequalities Internat.J.Math. and Math.Sci. 1999,**22**,535-542.

[7] Kuang Jichang, On new extensions of Hilbert's integral inequality, *J. Math. Anal. Appl.,* **235**(1999),608-614.

[8] Zhao Changjian, On Inverses of disperse and continuous Pachpatte's inequalities, *Acta Math. Sin.* **46**(2003),1111-1116.

[9] Zhao Changjian and L. Debnath, Some New Inverse Type Hilbert Integral Inequalities, *J. Math. Anal. Appl.,* 2001, **262**, 411–418.

[10] E. F. Beckenbach and R. Bellman, Inequalities, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1961.

[11] Zhao Changjian and Mihály, On the Inverse Inequalities of Two New Type Hilbert Integral Inequalities, *Octo. Math. Maga.,* 2003, **11**, 36-41.