On Refinements of Two New Integral Inequalities

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Abstract The main purpose of the present article is to generalize two new Hilbert type integral inequalities which is recent given by Pachpatte, and get two more wide results.

MR (2000) Subject Classification 26D15

Keywords Hilbert integral inequality, Hölder integral inequality.

1 Introduction

In resent years several authors[1−5] have given considerable attention to Hilbert integral inequalities and Hilbert's type integral inequalities and their various generalizations and applications. In 2000, Pachpatte [6] proved two new integral inequalities similar to certain extensions of Hilbert's integral inequality. In this paper we will generalize these two new inequalities.

2 Main Results

Our main results are given in the following theorems.

THEOREM 1 Let $h \geq 1$ and $l \geq 1$ be constants and $\frac{1}{p}+\frac{1}{q}=1, p>1$ and $I_{\alpha\beta}=(\alpha,\beta)$, Let $f(s)$ and $g(t)$ be real-valued continuous functions defined on I_{ax} and I_{by}, resectively, then

$$\int_a^x\int_b^y \frac{\mid F(s,h,a)\mid \cdot \mid G(t,l,b)\mid}{hl\Big(q(s-a)^{p-1}+p(t-b)^{q-1}\Big)}dsdt \leq K(p,q,x,y,a,b)\Big(\int_a^x (x-s)\mid f^{h-1}(s)f'(s)\mid^p ds\Big)^{1/p}$$

$$\times\Big(\int_b^y (y-t)\mid g^{l-1}(t)g'(t)\mid^q dt\Big)^{1/p} \tag{1}$$

where

$$K(p,q,x,y,a,b)=\frac{1}{pq}(x-a)^{(p-1)/p}(y-b)^{(q-1)/q}. \tag{2}$$

Proof: From the hypotheses, we have the following identities

$$F(s,h,a)=h\int_a^s f'(\tau)f^{h-1}(\tau)d\tau, s\in I_{ax}$$

where $F(s,h,a)=f^h(s)-f^h(a)$,

Foundation item. 1. Supported by National Natural Sciences Foundation of China (10271071).

2. Supported by Academic Mainstay of Middle-age and Youth Foundation of Shandong Province of China.

Hence

$$| F(s,h,a) |\leq h\int_a^s | f'(\tau)f^{h-1}(\tau) | d\tau$$

$$\leq h(s-a)^{(p-1)/p}\Big(\int_a^s | f'(\tau)f^{h-1}(\tau) |^p d\tau\Big)^{1/p} \tag{3}$$

Similarly,

$$| G(t,l,b) |\leq l(t-b)^{(q-1)/q}\Big(\int_b^t | g'(\sigma)g^{l-1}(\sigma) |^q d\sigma\Big)^{1/q} \tag{4}$$

By (3),(4) and the elementary inequality[7]

$$xy\leq\frac{x^p}{p}+\frac{y^q}{q}, \tag{5}$$

where $x\geq 0, y\geq 0$ and $\frac{1}{p}+\frac{1}{q}=1, p>1$, we have

$$| F(s,h,a) || G(t,l,b) |\leq hl\frac{q(s-a)^{(p-1)/p}+p(t-b)^{(q-1)/q}}{pq}\Big(\int_a^s | f'(\tau)f^{h-1}(\tau) |^p d\tau\Big)^{1/p}$$

$$\times\Big(\int_b^t | g'(\sigma)g^{l-1}(\sigma) |^q d\sigma\Big)^{1/q} \tag{6}$$

Dividing both sides of (6) by $hl\Big(q(s-a)^{(p-1)/p}+p(t-b)^{(q-1)/q}\Big)$ and then integrating first over t from b to y and integrating both sides of the resulting inequality over s from a to x and using Holder integral inequality[8], we have

$$\int_a^x\int_b^y\frac{| F(s,h,a) |\cdot| G(t,l,b) |}{hl\Big(q(s-a)^{p-1}+p(t-b)^{q-1}\Big)}dsdt\leq\frac{1}{pq}\int_a^x\Big(\int_a^s | f'(\tau)f^{h-1}(\tau) |^p d\tau\Big)^{1/p}ds$$

$$\times\int_b^y\Big(\int_b^t | g'(\sigma)g^{l-1}(\sigma) |^q d\sigma\Big)^{1/q}dt$$

$$\leq\frac{1}{pq}(x-a)^{(p-1)/p}(y-b)^{(q-1)/q}\Big(\int_a^x\Big(\int_a^s | f'(\tau)f^{h-1}(\tau) |^p d\tau\Big)ds\Big)^{1/p}$$

$$\times\Big(\int_b^y\Big(\int_b^t | g'(\sigma)g^{l-1}(\sigma) |^q d\sigma\Big)dt\Big)^{1/q}$$

$$\leq K(p,q,x,y,a,b)\Big(\int_a^x(x-s) | f^{h-1}(s)f'(s) |^p ds\Big)^{1/p}\Big(\int_b^y(y-t) | g^{l-1}(t)g'(t) |^q dt\Big)^{1/p}$$

The proof is complete.

Remark 1: Taking $h=l=1, a\to 0, b\to 0$ and $f(0)=g(0)=0$ in (1), then inequality (1) reduces to the following inequality

$$\int_0^x\int_0^y\frac{| f(s) |\cdot| g(t) |}{qs^{p-1}+pt^{q-1}}dsdt\leq K(p,q,x,y)\Big(\int_0^x(x-s) | f'(s) |^p ds\Big)^{1/p}$$

$$\Big(\int_0^y(y-t) | g'(t) |^q dt\Big)^{1/p}, \tag{7}$$

where $K(p,q,x,y)=\frac{1}{pq}x^{(p-1)/p}y^{(q-1)/q}$.

This is just an new inequality which was proved by Pachpatte[6].

THEOREM 2: Let $h \geq 1$ and $l \geq 1$ be constants and let $I_{\alpha\beta}$ be as in Theorem 1 and $\frac{1}{p}+\frac{1}{q}=1, p>1$. Let $f(s,t)$ and $g(k,r)$ be real-valued continuous functions defined on $I_{ax} \times I_{by}$ and $I_{cz} \times I_{dw}$, respectively. we denote the partial derivatives $(\partial/\partial s)u(s,t), (\partial/\partial t)u(s,t)$ and $(\partial^2/\partial s\partial t)u(s,t)$ by $D_1u(s,t), D_2u(s,t)$ and $D_2D_1u(s,t) = D_1D_2u(s,t)$, respectively , then

$$\int_a^x \int_b^y \left(\int_c^z \int_d^w \frac{\mid F(s,t,h,a,b) \mid \cdot \mid G(k,r,l,c,d) \mid}{q\Big((s-a)(t-b)\Big)^{p-1} + p\Big((k-c)(r-d)\Big)^{q-1}} dkdr \right) dsdt \leq C(p,q,x,y,z,w,a,b,c,d)$$

$$\times \left(\int_a^x \int_b^y (x-s)(y-t) \mid D_2^* D_1^* f(s,t,h) \mid^p dsdt \right)^{1/p}$$

$$\times \left(\int_c^z \int_d^w (z-k)(w-r) \mid D_2^* D_1^* g(k,r,l) \mid^q dkdr \right)^{1/q} \tag{8}$$

where

$$F(s,t,h,a,b) = f^h(s,t) - f^h(a,t) - f^h(s,b) + f^h(a,b),$$

$$D_2^* D_1^* f(s,t,h) = h(h-1)f^{h-1}(s,t) \cdot D_1 f(s,t) \cdot D_2 f(s,t) + hf^{h-1}(s,t) \cdot D_2 D_1 f(s,t),$$

$$G(k,r,l,c,d) = g^l(k,r) - g^l(c,r) - g^l(k,d) + g^l(c,d),$$

$$D_2^* D_1^* G(k,r,l) = l(l-1)g^{l-1}(k,r) \cdot D_1 g(k,r) \cdot D_2 g(k,r) + lg^{l-1}(k,r) \cdot D_2 D_1 g(k,r),$$

and

$$C(p,q,x,y,z,w,a,b,c,d) = \frac{1}{pq}\Big((x-a)(y-b)\Big)^{(p-1)/p}\Big((z-c)(w-d)\Big)^{(q-1)/q} \tag{9}$$

and $h \geq 1, l \geq 1$, a, b, c and d are constants.

Proof: From the hypotheses of Theorem 2, it is to note that

$$F(s,t,h,a,b) = \int_a^s \int_b^t D_2^* D_1^* f(\xi,\eta,h) d\xi d\eta, \tag{10}$$

where $(s,t) \in I_{ax} \times I_{by}$.

This is, since

$$\int_a^s \int_b^t D_2^* D_1^* f(\xi,\eta,h) d\xi d\eta$$

$$= \int_a^s \int_b^t \Big(h(h-1)f^{h-2}(\xi,\eta) \cdot D_1 f(\xi,\eta) \cdot D_2 f(\xi,\eta) + hf^{h-1}(\xi,\eta) \cdot D_2 D_1 f(\xi,\eta) \Big) d\xi d\eta$$

$$= \int_a^s \left(\int_b^t D_2 \Big(hf^{h-1}(\xi,\eta) \cdot D_1 f(\xi,\eta) \Big) d\eta \right) d\xi$$

$$= \int_a^s \Big(D_1 f^h(\xi,t) - D_1 f^h(\xi,b) \Big) d\xi = \int_a^s D_1 f^h(\xi,t) d\xi - \int_a^s D_1 f^h(\xi,b) d\xi$$

$$= f^h(s,t) - f^h(a,t) - f^h(s,b) + f^h(a,b) = F(s,t,h,a,b). \tag{11}$$

for (10), by applying Holder integral inequality

$$| F(s,t,h,a,b) | \leq \int_a^s \int_b^t | D_2^* D_1^* f(\xi,\eta,h) | d\xi d\eta$$

$$\leq \Big((s-a)(t-b)\Big)^{(p-1)/p} \Big(\int_a^s \int_b^t | D_2^* D_1^* f(\xi,\eta,h) |^p \, d\xi d\eta\Big)^{1/p} \tag{12}$$

Similarly,

$$| G(k,r,l,c,d) | \leq \Big((k-c)(r-d)\Big)^{(q-1)/q} \Big(\int_c^k \int_d^r | D_2^* D_1^* g(\sigma,\tau,l) |^q \, d\sigma d\tau\Big)^{1/q} \tag{13}$$

By (12),(13) and (5), we have

$$| F(s,t,h,a,b) | \cdot | G(k,r,l,c,d) | \leq \Big((s-a)(t-b)\Big)^{(p-1)/p} \Big(\int_a^s \int_b^t | D_2^* D_1^* f(\xi,\eta,h) |^p \, d\xi d\eta\Big)^{1/p}$$

$$\times \Big((k-c)(r-d)\Big)^{(q-1)/q} \Big(\int_c^k \int_d^r | D_2^* D_1^* g(\sigma,\tau,l) |^q \, d\sigma d\tau\Big)^{1/q}$$

$$\leq \frac{q\Big((s-a)(t-b)\Big)^{p-1} + p\Big((k-c)(r-d)\Big)^{q-1}}{pq} \Big(\int_a^s \int_b^t | D_2^* D_1^* f(\xi,\eta,h) |^p \, d\xi d\eta\Big)^{1/p}$$

$$\times \Big(\int_c^k \int_d^r | D_2^* D_1^* g(\sigma,\tau,l) |^q \, d\sigma d\tau\Big)^{1/q} \tag{14}$$

Dividing both sides of (14) by $q\Big((s-a)(t-b)\Big)^{p-1} + p\Big((k-c)(r-d)\Big)^{q-1}$ and then integrating first over r from d to w then over k from c to z and integrating both sides of the resulting inequality over t from b to y and over s from a to x and using Holder integral inequality and Fubini's Theorem[6], we have

$$\int_a^x \int_b^y \left(\int_c^z \int_d^w \frac{| F(s,t,h,a,b) | \cdot | G(k,r,l,c,d) |}{q\Big((s-a)(t-b)\Big)^{p-1} + p\Big((k-c)(r-d)\Big)^{q-1}} dk dr \right) ds dt$$

$$\leq \frac{1}{pq} \int_a^x \int_b^y \Big(\int_a^s \int_b^t | D_2^* D_1^* f(\xi,\eta,h) |^p \, d\xi d\eta\Big)^{1/p} ds dt$$

$$\times \int_c^z \int_d^w \Big(\int_c^k \int_d^r | D_2^* D_1^* g(\sigma,\tau,l) |^q \, d\sigma d\tau\Big)^{1/q} dk dr$$

$$\leq \frac{1}{pq} \Big((x-a)(y-b)\Big)^{(p-1)/p} \Big((k-c)(r-d)\Big)^{(q-1)/q} \left(\int_a^x \int_b^y \Big(\int_a^s \int_b^t | D_2^* D_1^* f(\xi,\eta,h) |^p \, d\xi d\eta\Big) ds dt \right)^{1/p}$$

$$\times \left(\int_c^z \int_d^w \Big(\int_c^k \int_d^r | D_2^* D_1^* g(\sigma,\tau,l) |^q \, d\sigma d\tau\Big) ds dt \right)^{1/q} dk dr$$

$$\leq C(p,q,x,y,z,w,a,b,c,d) \times \left(\int_a^x \int_b^y (x-s)(y-t) \, | D_2^* D_1^* f(s,t,h) |^p \, ds dt \right)^{1/p}$$

$$\times \Big(\int_c^z \int_d^w (z-k)(w-r) \, | D_2^* D_1^* g(k,r,l) |^q \, dk dr \Big)^{1/q}$$

The proof is complete.

Remark 2: It is obvious that inequality (8) is a new Hilbert's type inequality.

We take $h = l = 1, a \to 0, b \to 0, c \to 0, d \to 0, f(0,0) = f(0,t) + f(s,0)$ and $g(0,0) = g(0,r) + g(k,0)$ in (8), we obvious have $F(s,t,h,a,b) = f(s,t), G(k,r,l,c,d) = g(k,r), D_2^* D_1^* f(s,t,h) = D_2 D_1 f(s,t)$ and $D_2^* D_1^* g(k,r,l) = D_2 D_1 g(k,r)$, then inequality (8) reduces to the following inequality

$$\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{\mid f(s,t) \mid \cdot \mid g(k,r) \mid}{q(st)^{p-1} + p(kr)^{q-1}} dkdr \right) dsdt \le C(p,q,x,y,z,w)$$

$$\left(\int_0^x \int_0^y (x-s)(y-t) \mid D_2 D_1 f(s,t) \mid^p dsdt \right)^{1/p} \left(\int_0^z \int_0^w (z-k)(w-r) \mid D_2 D_1 g(k,r) \mid^q dkdr \right)^{1/q}$$

where

$$C(p,q,x,y,z,w) = \frac{1}{pq}(xy)^{(p-1)/p}(zw)^{(q-1)/q}.$$

This is just another new inequality which was proved by Pachpatte[6].

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