THE PROPERTIES OF THE GENERALIZED HERON MEAN AND ITS DUAL FORM

ZHI-HUA ZHANG, YU-DONG WU, AND AN-PING ZHAO

Abstract. In this paper, we define the generalized Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$, and obtain some propositions for the same means. In the final, an open problem is posed.

1. Introduction and Definition

For positive numbers a, b, let

$$G = G(a,b) = \sqrt{ab}; \tag{1.1}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{1.2}$$

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.3}$$

These are respectively called the geometric, logarithmic, and Heron means.

In 2003, Zh.-G. Xiao and Zh.-H. Zhang [1] gave the generalization of Heron mean and its dual form respectively as follows

$$H(a,b;k) = \frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \tag{1.4}$$

and

$$h(a,b;k) = \frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}}, \tag{1.5}$$

where k is a natural number. Authors proved that $H(a,b;k)$ is monotone decreasing function and $h(a,b;k)$ is monotone increasing function for k, and $\lim_{k\to+\infty} H(a,b;k) = \lim_{k\to+\infty} h(a,b;k) = L(a,b)$.

Let r be a real number, the r-order power mean (see [2]) is defined by

$$M_r = M_r(a,b) = \begin{cases} \left(\dfrac{a^r + b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.6}$$

Date: March 31, 2004.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D10.

Key words and phrases. Heron mean; Inequality; Monotonicity; Logarithmic Convexity; Proposition.

The authors would like to thank professor Wan-lan Wang and the anonymous referee for some valuable suggestions which have improved the final version of this paper.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

In [3], G. Jia and J.-D. Cao studied the power-type generalization of Heron mean

$$H_r = H_p(a,b) = \begin{cases} \left[\dfrac{a^r + (ab)^{r/2} + b^r}{3}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0; \end{cases} \tag{1.7}$$

and obtained inequalities

$$L \leqslant H_p \leqslant M_q, \tag{1.8}$$

where $p \geqslant \frac{1}{2}, q \geqslant \frac{2}{3}p$. Furthermore, $p = \frac{1}{2}, q = \frac{1}{3}$ are the best constants.

Combining (1.7), (1.4) and (1.5), two class of new means for two variables will be defined.

Definition 1.1. *Let $a > 0, b > 0,$ k is a natural number, and r is a real number, then the generalized power-type Heron mean and its dual form are defined as follows*

$$H_r(a,b;k) = \begin{cases} \left[\dfrac{1}{k+1}\sum_{i=0}^{k} a^{\frac{(k-i)r}{k}} b^{\frac{ir}{k}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0; \end{cases} \tag{1.9}$$

and

$$h_r(a,b;k) = \begin{cases} \left[\dfrac{1}{k}\sum_{i=1}^{k} a^{\frac{(k+1-i)r}{k+1}} b^{\frac{ir}{k+1}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.10}$$

According to Definition1.1, we easily find the following characteristic properties and two remarks for $H_r(a,b;k)$ and $h(a,b.;k)$.

Proposition 1.1. *If k is a natural number, and r is a real number, then*

(*a*) $H_r(a,b;k) = H_r(b,a;k),$ *and* $h_r(a,b;k) = h_r(b,a;k);$

(*b*) $\lim_{r\to 0} H_r(a,b;k) = \lim_{r\to 0} h_r(a,b;k) = G(a,b);$

(*c*) $H_r(a,b;1) = M_r(a,b),$ $H_r(a,b;2) = H_r(a,b),$ *and* $h_r(a,b;1) = G(a,b);$

(*d*) $\lim_{k\to+\infty} H_r(a,b;k) = \lim_{k\to+\infty} h_r(a,b;k) = [L(a^r,b^r)]^{\frac{1}{r}};$

(*e*) $a \leqslant H_r(a,b;k) \leqslant b,$ *and* $a \leqslant h_r(a,b;k) \leqslant b,$ *if* $0 < a < b;$

(*f*) $H_r(a,b;k) = h_r(a,b;k) = a$ *if and only if* $a = b;$

(*g*) $H_r(ta,tb;k) = tH_r(a,b;k),$ *and* $h_r(ta,tb;k) = th_r(a,b;k),$ *if* $t > 0.$

Remark 1.1. *Let $a > 0, b > 0,$ k is a natural number, and r is a real number, then the generalized power-type Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$ can be written that*

$$H_r(a,b;k) = \begin{cases} \left[\dfrac{a^{\frac{(k+1)r}{k}} - b^{\frac{(k+1)r}{k}}}{(k+1)(a^{\frac{r}{k}} - b^{\frac{r}{k}})}\right]^{\frac{1}{r}}, & r \neq 0, a \neq b; \\ \sqrt{ab}, & r = 0, a \neq b; \\ a, & r \in R, a = b; \end{cases} \tag{1.11}$$

or

$$H_r(a,b;k)=\begin{cases}\left[\int_0^1\left(xa^{\frac{r}{k}}+(1-x)b^{\frac{r}{k}}\right)^k dx\right]^{\frac{1}{r}}, & r\neq 0, a\neq b;\\ \sqrt{ab}, & r=0, a\neq b;\\ a, & r\in R, a=b;\end{cases} \tag{1.12}$$

and

$$h_r(a,b;k)=\begin{cases}\left[\frac{a^{\frac{kr}{k+1}}-b^{\frac{kr}{k+1}}}{-k(a^{-\frac{r}{k+1}}-b^{-\frac{r}{k+1}})}\right]^{\frac{1}{r}}, & r\neq 0, a\neq b;\\ \sqrt{ab}, & r=0, a\neq b;\\ a, & r\in R, a=b;\end{cases} \tag{1.13}$$

or

$$h_r(a,b;k)=\begin{cases}\left[\int_0^1\left(xa^{-\frac{r}{k+1}}+(1-x)b^{-\frac{r}{k+1}}\right)^{-k-1} dx\right]^{\frac{1}{r}}, & r\neq 0, a\neq b;\\ \sqrt{ab}, & r=0, a\neq b;\\ a, & r\in R, a=b.\end{cases} \tag{1.14}$$

Remark 1.2. *Let $a>0, b>0$, k is a natural number, then the following Detemple-Robertson mean $D_r(a,b)$ (see* [4]*) and its dual form $d_k(a,b)$ are respectively the special cases for $H_r(a,b;k)$ and $h_k(a,b;k)$:*

$$D_k(a,b)=[H_k(a,b;k)]^k=\frac{1}{k+1}\sum_{i=0}^{k}a^{k-i}b^k, \tag{1.15}$$

or

$$D_k(a,b)=\begin{cases}\frac{a^{k+1}-b^{k+1}}{(k+1)(a-b)}, & a\neq b;\\ a^k, & a=b;\end{cases} \tag{1.16}$$

$$d_k(a,b)=[h_{k+1}(a,b;k)]^{k+1}=\frac{1}{k}\sum_{i=1}^{k}a^{k+1-i}b^k, \tag{1.17}$$

or

$$d_k(a,b)=\begin{cases}\frac{ab(a^k-b^k)}{k(a-b)}, & a\neq b;\\ a^{k+1}, & a=b.\end{cases} \tag{1.18}$$

In this paper, we obtain the monotonicity and logarithmic convexity of the generalized power-type Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$. In the final, an open problem is posed.

2. Lemmas

In order to prove the theorems of the next section, we require some lemmas in this section.

Lemma 2.1. *([5],[6]) Let p, q be arbitrary real numbers, and $a, b > 0$. Then the extended mean values*

$$E_{p,q}(a,b) = \begin{cases} \left[\dfrac{q}{p}\cdot\dfrac{a^p-b^p}{a^q-b^q}\right]^{1/(p-q)}, & pq(p-q)(a-b)\neq 0; \\ \left[\dfrac{1}{p}\cdot\dfrac{a^p-b^p}{\ln a-\ln b}\right]^{1/p}, & p(a-b)\neq 0, q=0; \\ \dfrac{1}{e}\left(\dfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}, & p(a-b)\neq 0, p=q; \\ \sqrt{ab}, & (a-b)\neq 0, p=q=0; \\ a, & a=b. \end{cases} \tag{2.1}$$

are monotone increasing function with both p and q, or with both a and b; and are logarithmical concave on $(0,+\infty)$ with respect to either p or q, respectively; and logarithmical convex on $(-\infty,0)$ with respect to either p or q, respectively.

Lemma 2.2. *([7]) Let p, q, u, v be arbitrary with $p\neq q, u\neq v$. Then the inequality*

$$E_{p,q}(a,b)\geqslant E_{u,v}(a,b) \tag{2.2}$$

is satisfied for all $a, b > 0, a\neq b$ if and only if

$$p+q\geqslant u+v, \tag{2.3}$$

and

$$e(p,q)\geqslant e(u,v), \tag{2.4}$$

where

$$e(x,y) = \begin{cases} (x-y)/\ln(x/y), & for\ xy>0, x\neq y; \\ 0, & for\ xy=0; \end{cases} \tag{2.5}$$

if either $0\leqslant \min\{p,q,u,v\}$ or $\max\{p,q,u,v\}\leqslant 0$; and

$$e(x,y) = (\mid x\mid - \mid y\mid)/(x-y),\ \ for\ x,y\in \mathrm{R}, x\neq y, \tag{2.6}$$

if either $\min\{p,q,u,v\}<0<\max\{p,q,u,v\}$.

Lemma 2.3. *([2])Let $a_i, 1\leqslant i\leqslant n$ be real numbers with $a_i\neq a_j$ for $i\neq j$, and*

$$M_r(a) = \begin{cases} \left[\dfrac{1}{n}\displaystyle\sum_{i=1}^n a_i^r\right]^{\frac{1}{r}}, & 0<\mid r\mid<+\infty; \\ \displaystyle\prod_{i=1}^n a_i^{\frac{1}{n}}, & r=0. \end{cases} \tag{2.7}$$

Then $M_r(a)$ is monotone increasing function for r, and $f(r) = [M_r(a)]^r$ is logarithmic convex function with respect to $r>0$.

Lemma 2.4. *([8]) If $b_1\geqslant b_2\geqslant\cdots\geqslant b_n>0$, $\frac{a_1}{b_1}\geqslant\frac{a_2}{b_2}\geqslant\cdots\geqslant\frac{a_n}{b_n}>0$. Then the function*

$$F_r(a,b) = \begin{cases} \left[\displaystyle\sum_{i=1}^n a_i^r\Big/\sum_{i=1}^n b_i^r\right]^{\frac{1}{r}}, & r\neq 0, \\ \left(\displaystyle\prod_{i=1}^n \dfrac{a_i}{b_i}\right)^{1/n}, & r=0, \end{cases} \tag{2.8}$$

is monotone increasing one with respect to r.

Lemma 2.5. *If $x \geqslant 1$, and k is a fixed natural number. Then the functions*

$$f_k(x) = \left(\sum_{i=0}^{k} x^{k-i}\right)^{\frac{1}{k}} / \left(\sum_{i=0}^{k+1} x^{k+1-i}\right)^{\frac{1}{k+1}} \tag{2.9}$$

and

$$g_k(x) = \left(\sum_{i=1}^{k} x^{k+1-i}\right)^{\frac{1}{k+1}} / \left(\sum_{i=1}^{k+1} x^{k+2-i}\right)^{\frac{1}{k+2}} \tag{2.10}$$

both are monotone decreasing ones with respect to $x \in [1, +\infty)$.

Proof. Calculating the derivative for $f_k(x)$ and $g_k(x)$ about x, respectively, we get

$$f_k'(x) = \left[\sum_{i=1}^{k} \frac{i(i+1)}{2}(x^{i-1} - x^{2k-i})\right] / \left[k(k+1)\left(\sum_{i=0}^{k} x^{k-i}\right)^{\frac{k-1}{k}} \left(\sum_{i=0}^{k+1} x^{k+1-i}\right)^{\frac{k+2}{k+1}}\right],$$

and

$$g_k'(x) = \left[x\sum_{i=1}^{k} \frac{i(i+1)}{2}(x^{i-1} - x^{2k-i})\right] / \left[(k+1)(k+2)\left(\sum_{i=1}^{k} x^{k+1-i}\right)^{\frac{k}{k+1}} \left(\sum_{i=1}^{k+1} x^{k+2-i}\right)^{\frac{k+3}{k+2}}\right].$$

Since $x \geqslant 1$ and k is a fixed natural number, we find that $x^{i-1} - x^{2k-i} \leqslant 0, (1 \leqslant i \leqslant k)$, or $f_k'(x) \leqslant 0$ and $g_k'(x) \leqslant 0$, It is to see that the functions $f_k(x)$ and $g_k(x)$ both are monotone decreasing ones with respect to $x \in [1, +\infty)$. The proof of Lemma2.5 is completed. ∎

3. Monotonicity and Logarithmic Convexity

From Lemma2.1 and Lemma2.3, we easily prove the following Theorem3.1 and Theorem3.2, respectively.

Theorem 3.1. *If k is a fixed natural number, then $H_r(a,b;k)$ and $h_r(a,b;k)$ both are monotone increasing function with both a and b for fixed real numbers r, or with r for fixed positive numbers a and b; and are logarithmical concave on $(0,+\infty)$, and logarithmical convex on $(-\infty,0)$ with respect to r.*

Theorem 3.2. *Assume a and b are fixed positive numbers, and k is a fixed natural number, then $[H_r(a,b;k)]^r$ and $[h_r(a,b;k)]^r$ both are logarithmic convex function for $r > 0$.*

Theorem 3.3. [1] *For any $r > 0$, we have that $H_r(a,b;k)$ is monotonic decreasing function, and $h_r(a,b;k)$ is monotone increasing function with k.*

Theorem 3.4. *If $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, then $H_r(a_1,a_2;k)/H_r(b_1,b_2;k)$ and $h_r(a_1,a_2;k)/h_r(b_1,b_2;k)$ are monotone increasing functions with r on $\mathbf{R}$.*

Proof. According to Definition1.1, we have

$$\frac{H_r(a_1,a_2;k)}{H_r(b_1,b_2;k)} = \begin{cases} \left[\sum_{i=0}^{k} a_1^{\frac{(k-i)r}{k}} a_2^{\frac{ir}{k}} / \sum_{i=0}^{k} b_1^{\frac{(k-i)r}{k}} b_2^{\frac{ir}{k}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{\frac{a_1 a_2}{b_1 b_2}}, & r = 0. \end{cases} \tag{3.1}$$

and

$$\frac{h_r(a_1,a_2;k)}{h_r(b_1,b_2;k)}=\begin{cases}\left[\sum_{i=1}^{k}a_1^{\frac{(k+1-i)r}{k+1}}a_2^{\frac{ir}{k+1}}/\sum_{i=1}^{k}b_1^{\frac{(k+1-i)r}{k+1}}b_2^{\frac{ir}{k+1}}\right]^{\frac{1}{r}}, & r\neq 0;\\ \sqrt{\frac{a_1a_2}{b_1b_2}}, & r=0.\end{cases} \tag{3.2}$$

For $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, we find

$$b_1 \geqslant b_1^{\frac{k-1}{k}}b_2^{\frac{1}{k}} \geqslant b_1^{\frac{k-2}{k}}b_2^{\frac{2}{k}} \geqslant \cdots \geqslant b_2 > 0, \tag{3.3}$$

and

$$\frac{a_1}{b_1} \geqslant \left(\frac{a_1}{b_1}\right)^{\frac{k-1}{k}}\left(\frac{a_2}{b_2}\right)^{\frac{1}{k}} \geqslant \left(\frac{a_1}{b_1}\right)^{\frac{k-2}{k}}\left(\frac{a_2}{b_2}\right)^{\frac{2}{k}} \geqslant \cdots \geqslant \frac{a_2}{b_2} > 0. \tag{3.4}$$

From Lemma2.4, combining (3.1)-(3.4), the proof of Theorem 3.4 is completed. ∎

Theorem 3.5. *If* $0 < a \leqslant b \leqslant \frac{1}{2}$*, then* $H_r(a,b;k)/H_r(1-a,1-b;k)$ *and* $h_r(a,b;k)/h_r(1-a,1-b;k)$ *are monotone increasing functions for* r*.*

Proof. From $0 < a \leqslant b \leqslant \frac{1}{2}$, we get

$$0 < 1-a \leqslant 1-b, \text{ and } 0 < \frac{a}{1-a} \leqslant \frac{b}{1-b}. \tag{3.5}$$

Using Theorem3.4, we obtain Theorem 3.5. ∎

Theorem 3.6. *If* $b_1 \geqslant b_2 > 0$ *and* $a_1/b_1 \geqslant a_2/b_2 > 0$*, then* $(D_k(a_1,a_2)/D_k(b_1,b_2))^{\frac{1}{k}}$ *and* $(d_k(a_1,a_2)/d_k(b_1,b_2))^{\frac{1}{k+1}}$ *both are monotone increasing functions with* k *on* $\mathbf{N}$*.*

Proof. To prove $(D_k(a_1,a_2)/D_k(b_1,b_2))^{\frac{1}{k}}$ is monotone increasing function with k on $\mathbf{N}$, we only want to prove that: if $b_1 \geqslant b_2 > 0$, $a_1/b_1 \geqslant a_2/b_2 > 0$ and k is a natural number, then

$$\left(\sum_{i=0}^{k}a_1^{k-i}a_2^{i}/\sum_{i=0}^{k}b_1^{k-i}b_2^{i}\right)^{\frac{1}{k}} \leqslant \left(\sum_{i=0}^{k+1}a_1^{k+1-i}a_2^{i}/\sum_{i=0}^{k+1}b_1^{k+1-i}b_2^{i}\right)^{\frac{1}{k+1}}, \tag{3.6}$$

or

$$\left[\sum_{i=0}^{k}\left(\frac{a_1}{a_2}\right)^{k-i}\right]^{\frac{1}{k}} / \left[\sum_{i=0}^{k+1}\left(\frac{a_1}{a_2}\right)^{k+1-i}\right]^{\frac{1}{k+1}} \leqslant \left[\sum_{i=0}^{k}\left(\frac{b_1}{b_2}\right)^{k-i}\right]^{\frac{1}{k}} / \left[\sum_{i=0}^{k+1}\left((\frac{b_1}{b_2}\right)^{k+1-i}\right]^{\frac{1}{k+1}}. \tag{3.7}$$

Taking $x_1 = \frac{a_1}{a_2}, x_2 = \frac{b_1}{b_2}$, we have $x_1 \geqslant x_2 \geqslant 1$, and inequality (3.7) is equivalent to

$$\left(\sum_{i=0}^{k}x_1^{k-i}\right)^{\frac{1}{k}} / \left(\sum_{i=0}^{k+1}x_1^{k+1-i}\right)^{\frac{1}{k+1}} \leqslant \left(\sum_{i=0}^{k}x_2^{k-i}\right)^{\frac{1}{k}} / \left(\sum_{i=0}^{k+1}x_2^{k+1-i}\right)^{\frac{1}{k+1}}. \tag{3.8}$$

From Lemma2.5, we find (3.8) or (3.6).

By the same way, we can prove that $(d_k(a_1,a_2)/d_k(b_1,b_2))^{\frac{1}{k+1}}$ is monotone increasing function with k on $\mathbf{N}$. Thus, Theorem3.6 is proved. ∎

The above-hand of Theorem3.6 is obtained by W.-L. Wang, G.-X. Li and J. Chen in 1988 (see [9]). By the same way of the proof of Theorem3.5, we can obtain

Theorem 3.7. *If* $0 < a \leqslant b \leqslant \frac{1}{2}$*, then* $(D_k(a,b)/D_k(1-a,1-b))^{\frac{1}{k}}$ *and* $(h_k(a,b)/h_k(1-a,1-b))^{\frac{1}{k+1}}$ *both are monotone increasing functions for* r*.*

Remark 3.1. *Let $k \to +\infty$, from Proposition1.1 (d), we have*

$$\lim_{k\to+\infty} h_r(a,b;k) = \lim_{k\to+\infty} H_r(a,b;k) = [L(a^r,b^r)]^{\frac{1}{r}}. \tag{3.9}$$

According to some theorems above, we immediately get some similar results with $[L(a^r,b^r)]^{\frac{1}{r}}$:

(a) $[L(a^r,b^r)]^{\frac{1}{r}}$ *are monotone increasing function with both a and b for fixed real numbers r, or with r for fixed positive numbers a and b; and are logarithmical concave on $(0,+\infty)$ with respect to r; and logarithmical convex on $(-\infty,0)$ with respect to r;*

(b) *Assume a and b are fixed positive numbers, then $L(a^r,b^r)$ is logarithmic convex function for $r>0$;*

(c) *If $b_1 \geqslant b_2 > 0$ and $a_1/b_1 \geqslant a_2/b_2 > 0$, then $[L(a_1^r,a_2^r)/L(b_1^r,b_2^r)]^{\frac{1}{r}}$ is monotone increasing function with r on $\mathbf{R}$;*

(d) *If $0 < a \leqslant b \leqslant \frac{1}{2}$, then $[L(a^r,b^r)/L((1-a)^r,(1-b)^r)]^{\frac{1}{r}}$ is monotone increasing function for $r \in \mathbf{R}$.*

4. Some Inequalities

Theorem 4.1. *Let k_1, k_2 are two fixed natural numbers. If $r>0$, we then have inequality*

$$h_r(a,b;k_1) \leqslant H_r(a,b;k_2), \tag{4.1}$$

and inverse inequality holds if $r<0$. With equality holding if and only if $a=b$.

Proof. If $r>0$, from Remaek1.1, that (4.1) is equivalent to

$$\left[\frac{a^{\frac{k_1 r}{k_1+1}} - b^{\frac{k_1 r}{k_1+1}}}{-k_1(a^{-\frac{r}{k_1+1}} - b^{-\frac{r}{k_1+1}})}\right]^{\frac{1}{r}} \leqslant \left[\frac{a^{\frac{(k_2+1)r}{k_2}} - b^{\frac{(k_2+1)r}{k_2}}}{(k_2+1)(a^{\frac{r}{k_2}} - b^{\frac{r}{k_2}})}\right]^{\frac{1}{r}}. \tag{4.2}$$

Setting $p=\frac{(k_2+1)r}{k_2}, q=\frac{r}{k_2}, u=\frac{k_1 r}{k_1+1}$, and $v=-\frac{r}{k_1+1}$, that (4.2) become

$$E_{p,q}(a,b) \geqslant E_{u,v}(a,b). \tag{4.3}$$

For k_1, k_2 are two fixed natural numbers, that is easy to see that

$$\min\{p,q,u,v\} = -\frac{r}{k_1+1} < 0 < \max\{p,q,u,v\}, \tag{4.4}$$

$$p+q = \frac{(k_2+2)r}{k_2} > \frac{(k_1-1)r}{k_1+1} = u+v. \tag{4.5}$$

and

$$e(p,q) = r > \frac{(k_1-1)r}{k_1+1} = e(u,v), \tag{4.6}$$

where $e(x,y)$ is defined as (2.6) of Lemma2.2.

Using Lemma2.2, and combining expression (4.4)-(4.6), we can obtain (4.3), and immediately follow that expression (4.1) is true. Thus, the proof of Theorem4.1 is completed. ∎

By the same way , we can obtain

Theorem 4.2. *Let k be a fixed natural number. We then have inequality*

$$(d_k(a,b))^{\frac{1}{k+1}} \leqslant (D_k(a,b))^{\frac{1}{k}}, \tag{4.7}$$

with equality holding if and only if $a=b$.

Combining Theorem4.1, Proposition1.1 (d) and Theorem3.3, we get

Corollary 4.1. *If $r_1 < 1 < r_2$, and k_1, k_2 are two fixed natural numbers, then we have*

$$h_{r_1}(a,b;k_1) \leqslant L(a,b) \leqslant H_{r_2}(a,b;k_2), \tag{4.8}$$

with equalities holding if and only if $a = b$.

Remark 4.1. *From those theorems of the last section, for some special cases with k or r, we can obtain some inequalities.*

In the final, we put forward an open problem

Open Problem 4.1. *Prove that, if k_1, k_2 are two fixed natural number, and $p \geqslant \frac{k_1}{k_1+2}, q \geqslant \frac{(k_1+2)p}{3k_1}, 0 \leqslant r \leqslant \frac{k_2+1}{k_2-1}$, then the following inequalities for the new bounds of the logarithmic mean*

$$G(a,b) \leqslant h_r(a,b;k_2) \leqslant L(a,b) \leqslant H_p(a,b;k_1) \leqslant M_q(a,b).$$

hold, and the constants $p = \frac{k_1}{k_1+2}, q = \frac{1}{3}$, and $r = \frac{k_2+1}{k_2-1}$ $(k_2 > 1)$ are the best possible.

References

[1] Zh.-G. Xiao and Zh.-H. Zhang, *The Inequalities $G \leqslant L \leqslant I \leqslant A$ in n Variables*, J. Ineq. Pure & Appl. Math., **4**(2) (2003), Article 39. http://jipam.vu.edu.au/v4n2/110_02.pdf

[2] J.-Ch. Kuang, *Applied Inequalities*, Hunan Eduation Press, 2nd. Ed., 1993. (Chinese)

[3] G. Jia and J.-D. Cao, *A New Upper Bound of the Logarithmic Mean*. J. Ineq. Pure & Appl. Math., **4**(4) (2003), Article 80. http://jipam.vu.edu.au/v4n4/088_03.pdf.

[4] D. W. Detemple and J. M. Robertson, *On Generalized Symmetric Means of Two Varibles*, Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.634–672(1979), 236–238(MR, 81h:26014).

[5] F. Qi, *Logarithmic convexities of the Extended Mean Values*, RGMIA Resarch Report Collection **5**(2) (1999), Article5. http://rgmia.vu.edu.au.

[6] E. Lenach and M. Sholander, *Extended Mean Values*, Amer. Math. Monthly **85** (1978), 84–90.

[7] Zs. Páles, *Inequalities for Differences of Powers*, J. Math. Anal. & Appl., **131** (1988), 271-281.

[8] A. W. Marsall, I. Olkin and F. Proschan, *Monotonicty of Ratios of Means and other Applications of Majorization,* In Inequalities, edited by O.Shisha. New York London 1967, 177–190.

[9] W.-L. Wang, G.-X. Li and J. Chen, *Inequalities Involving Ratios of Means*, Journal of Chendu University of Science and Technology, **42** No.6 (1988), 83–88. (Chinese)

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400, P. R. China.
E-mail address: zxzh1234@163.com

(Y.-D. Wu) Xinchang Middle School, Xinchang, Zhejiang 312500, P. R. China.
E-mail address: zjxcwyd@tom.com

(An-P. Zhao) No.1 Middle School of Zixin, Chenzhou, Hunan 423406, P. R. China.