

A FURTHER GENERALIZATION OF HARDY-HILBERT'S INTEGRAL INEQUALITY WITH PARAMETER AND APPLICATIONS

HE LEPING, SEVER S. DRAGOMIR, AND YANG QIAOSHUN

ABSTRACT. In this paper, by introducing some parameters and by employing a sharpening of Hölder's inequality, a new generalization of Hardy-Hilbert integral inequality involving the Beta function is established. At the same time, an extension of Widder's theorem is given.

1. INTRODUCTION

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f, g : (0, \infty) \rightarrow (0, \infty)$ are so that

$$0 < \int_0^\infty f^p(t)dt < \infty, \quad 0 < \int_0^\infty g^q(t)dt < \infty.$$

Then we may state the following integral inequality

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)dt \right)^{\frac{1}{q}},$$

in which the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

The inequality (1.1) is well known in the literature as the Hardy-Hilbert's integral inequality.

Recently, some improvements and generalizations of Hardy-Hilbert's integral inequality have been given. For instance, we refer the reader to the papers [2]–[7] and the bibliography therein.

The main purpose of this paper is to establish a new extended Hardy-Hilbert's type inequality, which includes improvements and generalisations of the corresponding results from [2]–[3].

2. LEMMAS AND THEIR PROOFS

For convenience, we firstly introduce some notations:

$$(f^r, g^s) = \int_\alpha^\infty f^r(x)g^s(x)dx, \quad \|f\|_p = \left(\int_\alpha^\infty f^p(x)dx \right)^{\frac{1}{p}},$$

$$\|f\|_2 = \|f\|, \quad S_r(H, x) = \left(H^{r/2}, x \right) \|H\|_r^{-r/2},$$

where x is a parametric variable unit vector. Clearly, $S_r(H, x) = 0$ when the vector x selected is orthogonal to $H^{\frac{r}{2}}$.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hardy-Hilbert's integral inequality, Hölder's inequality, Weight coefficient, Beta function, Widder's theorem.

Throughout this paper, m is taken to be $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

In order to state our results, we need to point out the following lemmas.

Lemma 1. *Let $f(x), g(x) > 0, x \in (0, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < \infty, 0 < \|g\|_q < \infty$, then*

$$(2.1) \quad (f, g) < \|f\|_p \|g\|_q (1 - R)^m,$$

where $R = (S_p(f, h) - S_q(g, h))^2, \|h\| = 1, f^{p/2}(x), g^{q/2}(x)$ and $h(x)$ are linearly independent.

The lemma is proved in [4], and we omit the details.

In the following, we define

$$k_\lambda = B\left(\frac{p + \lambda - 2}{p}, \frac{q + \lambda - 2}{q}\right),$$

$$\theta_\lambda(r) = \int_0^1 \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du \quad (r = p, q),$$

where

$$B(u, v) = \int_0^1 \frac{t^{(-1+u)}}{(1+t)^{u+v}} dt \quad (u, v > 0)$$

is the *Beta function*.

The following lemma also holds.

Lemma 2. *Let $b < 1, \lambda > 0$. Define the function*

$$\varphi(b, y) = y^{-1+b} \int_0^y \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^b du, \quad y \in (0, 1].$$

Then we have

$$(2.2) \quad \varphi(b, y) > \varphi(b, 1), \quad (0 < y < 1).$$

A proof of Lemma 2 is given in paper [5], and we omit it here.

Another technical result that will be required in the following is:

Lemma 3. *Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 2 - \min\{p, q\}, \alpha \geq -\beta$. Define the weight function ω_λ by*

$$(2.3) \quad \omega_\lambda(\alpha, \beta, r, x) = \int_\alpha^\infty \frac{1}{(x+y+2\beta)^\lambda} \left(\frac{x+\beta}{y+\beta}\right)^{\frac{(2-\lambda)}{r}} dy \quad x \in (\alpha, \infty).$$

(i) For $\alpha = -\beta$,

$$(2.4) \quad \omega_\lambda(-\beta, \beta, r, x) = k_\lambda (x + \beta)^{1-\lambda} \quad x \in (-\beta, \infty).$$

(ii) For $\alpha > -\beta$,

$$(2.5) \quad \omega_\lambda(\alpha, \beta, r, x) < \left[k_\lambda - \theta_\lambda(r) \left(\frac{\alpha + \beta}{x + \beta}\right)^{1 + \frac{(\lambda-2)}{r}} \right] (x + \beta)^{1-\lambda} \quad x \in (\alpha, \infty).$$

Proof. Setting $u = (y + \beta)/(x + \beta)$, we have

$$\omega_\lambda(\alpha, \beta, r, x) = (x + \beta)^{1-\lambda} \int_{\frac{\alpha+\beta}{x+\beta}}^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du.$$

- (i) For $\alpha = -\beta$, (2.4) is valid.
(ii) For $\alpha > -\beta$, we have

$$\begin{aligned}
& \omega_\lambda(\alpha, \beta, r, x) \\
(2.6) \quad &= (x + \beta)^{1-\lambda} \left\{ \int_0^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du \right. \\
& \quad \left. - \int_0^{\frac{\alpha+\beta}{x+\beta}} \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{(2-\lambda)}{r}} du \right\} \\
(2.7) \quad &= (x + \beta)^{1-\lambda} \left\{ k_\lambda - \left(\frac{\alpha + \beta}{x + \beta}\right)^{1+\frac{(\lambda-2)}{r}} \varphi\left(\frac{2-\lambda}{r}, \frac{\alpha + \beta}{x + \beta}\right) \right\}.
\end{aligned}$$

Putting $b = \frac{2-\lambda}{r}$, and since $\lambda > 2 - \min\{p, q\}$, $b < 1$ is valid, then by Lemma 2 we get

$$(2.8) \quad \varphi\left(\frac{2-\lambda}{r}, \frac{\alpha + \beta}{x + \beta}\right) > \varphi\left(\frac{2-\lambda}{r}, 1\right) = \theta_\lambda(r) \quad (x \in (0, \infty)).$$

Substituting (2.8) into (2.7), we obtain (2.5). The proof is completed.

■

Finally, the following result is needed as well.

Lemma 4. *Let $a_n (n = 0, 1, 2, 3, \dots)$ be complex numbers. If*

$$A(z) := \sum_{n=0}^{\infty} a_n z^n$$

is analytic on unit disk $|z| \leq 1$, and

$$A^*(z) := \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is analytic on $|z| < \infty$, then

$$(2.9) \quad \int_0^1 |A(x)|^2 dx = \int_0^1 \left| \int_0^\infty e^{-s/x} A^*(s) ds \right|^2 \frac{1}{x^2} dx,$$

where $s \in (0, \infty)$, $x \in (0, 1]$.

Proof. Since $A^*(z)$ is analytic on the complex plane, the series

$$\sum_{n=0}^{\infty} \frac{e^{-t} a_n (xt)^n}{n!}$$

is uniformly convergent in $(0, \infty)$, and we obtain

$$\begin{aligned} \int_0^\infty e^{-t} A^*(tx) dt &= \int_0^\infty e^{-t} \sum_{n=0}^\infty \frac{a_n (xt)^n}{n!} dt \\ &= \sum_{n=0}^\infty \frac{a_n x^n}{n!} \int_0^\infty t^n e^{-t} dt \\ &= \sum_{n=0}^\infty a_n x^n = A(x). \end{aligned}$$

Setting $tx = s$, then

$$A(x) = \frac{1}{x} \int_0^\infty e^{-s/x} A^*(s) ds$$

whence

$$\int_0^1 |A(x)|^2 dx = \int_0^1 \left| \int_0^\infty e^{-s/x} A^*(s) ds \right|^2 \frac{1}{x^2} dx.$$

The lemma is thus proved. ■

3. MAIN RESULTS

For the sake of convenience, we need the following notations:

$$\begin{aligned} F(x, y) &= \frac{f(x)}{(x+y+2\beta)^{\lambda/p}} \left(\frac{x+\beta}{y+\beta} \right)^{\frac{(2-\lambda)}{pq}}, \\ G(x, y) &= \frac{g(y)}{(x+y+2\beta)^{\lambda/q}} \left(\frac{y+\beta}{x+\beta} \right)^{\frac{(2-\lambda)}{pq}}, \\ \phi(r, x) &= \int_0^{\frac{\alpha+\beta}{x+\beta}} \frac{1}{(1+u)^\lambda} \left(\frac{1}{u} \right)^{(2-\lambda)/r} du, \end{aligned} \quad (3.1)$$

$$S_p(F, h) = \left\{ \int_\alpha^\infty \int_\alpha^\infty F^{p/2} h dx dy \right\} \left\{ \int_\alpha^\infty [k_\lambda - \phi(q, x)] (x+\beta)^{1-\lambda} f^p(x) dx \right\}^{-\frac{1}{2}},$$

and

$$S_q(G, h) = \left\{ \int_\alpha^\infty \int_\alpha^\infty G^{q/2} h dx dy \right\} \left\{ \int_\alpha^\infty [k_\lambda - \phi(p, x)] (x+\beta)^{1-\lambda} g^q(x) dx \right\}^{-\frac{1}{2}},$$

where $h = h(x, y)$ is a unit vector satisfying the property

$$\|h\| = \left\{ \int_\alpha^\infty \int_\alpha^\infty h^2(x, y) dx dy \right\}^{\frac{1}{2}} = 1$$

and $F^{p/2}, G^{q/2}, h$ are linearly independent.

The first main result is incorporated in the following theorem.

Theorem 1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $\alpha \geq -\beta$, and $f, g > 0$. Assume also that*

$$0 < \int_\alpha^\infty (t+\beta)^{1-\lambda} f^p(t) dt < \infty,$$

and

$$0 < \int_\alpha^\infty (t+\beta)^{1-\lambda} g^q(t) dt < \infty.$$

(i) If $\alpha > -\beta$, then we have

$$(3.2) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y+2\beta)^{\lambda}} dx dy \\ < \left\{ \int_{\alpha}^{\infty} \left(k_{\lambda} - \theta_{\lambda}(q) \left(\frac{\alpha+\beta}{t+\beta} \right)^{1+(\lambda-2)/q} \right) (t+\beta)^{1-\lambda} f^p(t) dt \right\}^{\frac{1}{p}} \\ \times \left\{ \int_{\alpha}^{\infty} \left(k_{\lambda} - \theta_{\lambda}(p) \left(\frac{\alpha+\beta}{t+\beta} \right)^{1+(\lambda-2)/p} \right) (t+\beta)^{1-\lambda} g^q(t) dt \right\}^{\frac{1}{q}} (1-R_{\lambda})^m.$$

(ii) If $\alpha = -\beta$, then we have

$$(3.3) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y+2\beta)^{\lambda}} dx dy \\ < k_{\lambda} \left(\int_{-\beta}^{\infty} (t+\beta)^{1-\lambda} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{-\beta}^{\infty} (t+\beta)^{1-\lambda} g^q(t) dt \right)^{\frac{1}{q}} (1-\bar{R}_{\lambda})^m,$$

where

$$R_{\lambda} = (S_p(F, h) - S_q(G, h))^2,$$

while the function h is defined by

$$(3.4) \quad h(x, y) = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{e^{\alpha-x}}{(x+y-2\alpha)^{\frac{1}{2}}} \left(\frac{x-\alpha}{y-\alpha} \right)^{\frac{1}{4}}.$$

Proof. By Lemma 1 and the equality (2.3), we have

$$(3.5) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)}{(x+y+2\beta)^{\lambda}} dx dy \\ = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} FG dx dy \\ \leq \left\{ \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} F^p dx dy \right\}^{\frac{1}{p}} \left\{ \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} G^q dx dy \right\}^{\frac{1}{q}} (1-R_{\lambda})^m \\ = \left(\int_{\alpha}^{\infty} \omega_{\lambda}(\alpha, \beta, q, t) f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{\alpha}^{\infty} \omega_{\lambda}(\alpha, \beta, p, t) g^q(t) dt \right)^{\frac{1}{q}} (1-R_{\lambda})^m.$$

Substituting (2.5) and (2.4) into the inequality (3.5) respectively, the inequalities (3.2) and (3.3) follow.

Next, let us discuss the expression R_{λ} .

We can choose the function h indicated by (3.4). Setting $s = x - \alpha$ and $t = y - \alpha$, we get

$$\|h\|^2 = \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} h^2(x, y) dx dy = \frac{2}{\pi} \int_0^{\infty} e^{-2s} ds \int_0^{\infty} \frac{1}{s+t} \left(\frac{s}{t} \right)^{\frac{1}{2}} dt = 1.$$

Hence, $\|h\| = 1$.

By Lemma 1 and the given h , we have

$$(3.6) \quad R_\lambda = \left\{ \left(\int_\alpha^\infty \int_\alpha^\infty F^{p/2} h \, dx dy \right) \left(\int_\alpha^\infty \int_\alpha^\infty F^p \, dx dy \right)^{-\frac{1}{2}} \right. \\ \left. - \left(\int_\alpha^\infty \int_\alpha^\infty G^{q/2} h \, dx dy \right) \left(\int_\alpha^\infty \int_\alpha^\infty G^q \, dx dy \right)^{-\frac{1}{2}} \right\}^2.$$

Substituting (2.3), (2.6) and (3.1) into (3.6), we get

$$R_\lambda = (S_p(F, h) - S_q(G, h))^2.$$

It is obvious that $F^{p/2}$, $G^{q/2}$ and h are linearly independent, so it is impossible for equality to hold in (3.5).

The proof is thus completed. ■

Owing to $p, q > 1$, when $\lambda = 1, 2$; the condition $\lambda > 2 - \min\{p, q\}$ is satisfied. We have

$$\theta_1(r) = \int_0^1 \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1}{r}} du > \int_0^1 \frac{1}{1+u} du = \ln 2, \quad k_1 = B\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{\pi}{\sin(\pi/p)},$$

$$\theta_2(r) = \int_0^1 \frac{1}{(1+u)^2} du = \frac{1}{2}, \quad k_2 = B\left(\frac{p+2-2}{p}, \frac{q+2-2}{q}\right) = B(1, 1) = 1.$$

The following results are natural consequences of Theorem 1.

Corollary 1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq \beta$, $f, g > 0$,*

$$0 < \int_\alpha^\infty f^p(t) dt < \infty,$$

and

$$0 < \int_\alpha^\infty g^q(t) dt < \infty,$$

then

$$(3.7) \quad \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{x+y+2\beta} dx dy \\ < \left\{ \int_\alpha^\infty \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\alpha+\beta}{t+\beta} \right)^{\frac{1}{p}} \cdot \ln 2 \right) f^p(t) dt \right\}^{\frac{1}{p}} \\ \times \left\{ \int_\alpha^\infty \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\alpha+\beta}{t+\beta} \right)^{\frac{1}{q}} \cdot \ln 2 \right) g^q(t) dt \right\}^{\frac{1}{q}} (1 - R_1)^m,$$

$$(3.8) \quad \int_{-\beta}^\infty \int_{-\beta}^\infty \frac{f(x)g(y)}{x+y+2\beta} dx dy \\ < \frac{\pi}{\sin(\pi/p)} \left(\int_{-\beta}^\infty f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{-\beta}^\infty g^q(t) dt \right)^{\frac{1}{q}} (1 - \bar{R}_1)^m,$$

and

$$(3.9) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t) dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t) dt \right)^{\frac{1}{q}} (1-r_1)^m.$$

Remark 1. When $p = q = 2$, the inequality (3.9) reduces, after some simple computation, to an inequality obtained in [2].

Corollary 2. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq \beta$, $f, g > 0$,

$$0 < \int_\alpha^\infty (t+\beta)^{-1} f^p(t) dt < \infty$$

and

$$0 < \int_\alpha^\infty (t+\beta)^{-1} g^q(t) dt < \infty,$$

then

$$(3.10) \quad \int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x+y+2\beta)^2} dx dy < \left\{ \int_\alpha^\infty \left(1 - \frac{\alpha+\beta}{2(t+\beta)} \right) \frac{1}{t+\beta} f^p(t) dt \right\}^{\frac{1}{p}} \times \left\{ \int_\alpha^\infty \left(1 - \frac{\alpha+\beta}{2(t+\beta)} \right) \frac{1}{t+\beta} g^q(t) dt \right\}^{\frac{1}{q}} (1-R_2)^m$$

and

$$(3.11) \quad \int_{-\beta}^\infty \int_{-\beta}^\infty \frac{f(x)g(y)}{(x+y+2\beta)^2} dx dy < \left(\int_{-\beta}^\infty \frac{1}{t+\beta} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{-\beta}^\infty \frac{1}{t+\beta} g^q(t) dt \right)^{\frac{1}{q}} (1-\bar{R}_2)^m.$$

Remark 2. The inequalities (3.2), (3.3) and (3.7) – (3.9) are generalizations of (1.1).

Remark 3. We can also define $h(x, y)$ as

$$h(x, y) = \begin{cases} 1 & (x, y) \in [0, 1] \times [0, 1] \\ 0 & (x, y) \in (0, \infty) \times (0, \infty) \setminus [0, 1] \times [0, 1]. \end{cases}$$

In this case, the expression of R_λ will be much simpler. The details are omitted.

4. APPLICATIONS

We start with the following result:

Theorem 2. Suppose that $a_n (n = 0, 1, 2, 3, \dots)$ are complex numbers. Also, define $A(x) = \sum_{n=0}^\infty a_n x^n$, $A^*(x) = \sum_{n=0}^\infty \frac{a_n x^n}{n!}$, and the function f as:

$$(4.1) \quad f(x) = e^{-x} A^*(x), \quad x \in (0, \infty).$$

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(4.2) \quad \int_0^1 |A(x)|^2 dx < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_0^\infty |f(x)|^q dx \right)^{\frac{1}{q}} (1 - \bar{R})^m,$$

where

$$\bar{R} := (S_p(\bar{F}, h) - S_q(\bar{G}, h))^2 > 0,$$

with $\|h\| = 1$, and

$$\bar{F} := \frac{|f(s)|}{(s+t)^{\frac{1}{p}}} \left(\frac{s}{t} \right)^{\frac{1}{pq}},$$

$$\bar{G} := \frac{|f(t)|}{(s+t)^{\frac{1}{q}}} \left(\frac{t}{s} \right)^{\frac{1}{pq}}; \quad \psi(t) := \int_0^\infty \frac{e^{-s}}{(s+t)} \left(\frac{t}{s} \right)^{\frac{(q-p)}{2pq}} ds,$$

$$S_p(\bar{F}, h) := \sqrt{2} \left\{ \int_0^\infty e^{-s} |f(s)|^{p/2} ds \right\} \cdot \left\{ \int_0^\infty |f(s)|^p ds \right\}^{-\frac{1}{2}},$$

$$S_q(\bar{G}, h) := \frac{\sqrt{2} \sin(\pi/p)}{\pi} \left\{ \int_0^\infty \psi(t) |f(t)|^{q/2} dt \right\} \cdot \left\{ \int_0^\infty |f(s)|^q ds \right\}^{-\frac{1}{2}}$$

and

$$(\bar{F})^{p/2}, (\bar{G})^{q/2}, h$$

are linearly independent.

Proof. Setting $y = \frac{1}{x}$ on the right-hand side of the equality (2.9), we have

$$(4.3) \quad \int_0^1 |A(x)|^2 dx = \int_1^\infty \left| \int_0^\infty e^{-sy} A^*(s) ds \right|^2 dy.$$

Next, put $u = y - 1$. According to the equalities (4.1) and (4.3), we get

$$\int_0^1 |A(x)|^2 dx = \int_0^\infty du \left| \int_0^\infty e^{-su} f(s) ds \right|^2.$$

Using Hardy's technique, we may state that

$$\begin{aligned} \int_0^1 |A(x)|^2 dx &= \int_0^\infty du \left| \int_0^\infty e^{-su} f(s) ds \right|^2 \\ &= \int_0^\infty du \int_0^\infty e^{-su} f(s) ds \int_0^\infty e^{-tu} f(t) dt \\ &\leq \int_0^\infty \int_0^\infty \left(\int_0^\infty e^{-(s+t)u} du \right) |f(s)| |f(t)| ds dt \\ &= \int_0^\infty \int_0^\infty \frac{|f(s)| |f(t)|}{s+t} ds dt \\ &< \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty |f(x)|^p \right)^{\frac{1}{p}} \left(\int_0^\infty |f(x)|^q \right)^{\frac{1}{q}} (1 - \bar{R})^m. \end{aligned}$$

Let us choose the function $h(s, t)$ to be defined by

$$h(s, t) := \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{e^{-s}}{(s+t)^{\frac{1}{2}}} \left(\frac{s}{t} \right)^{\frac{1}{4}},$$

then

$$\|h\| = \left\{ \int_0^\infty \int_0^\infty h^2(s, t) ds dt \right\}^{\frac{1}{2}} = 1.$$

Notice that $k_1(p) = B(\frac{1}{q}, \frac{1}{p}) = \pi / \sin(\frac{\pi}{p})$, and, in a similar way to the one in Theorem 1, the expression of \bar{R} is easily given. We omit the details. ■

Remark 4. In particular, when $p = q = 2$, it follows from (4.2) that

$$(4.4) \quad \int_0^1 A^2(x) dx = \pi(1-r)^{\frac{1}{2}} \int_0^\infty f^2(x) dx.$$

If r in (4.4) is replaced by zero, then Widder's theorem (see [8]) can be recaptured.

Remark 5. After simple computation, the inequality (4.4) is equivalent to the inequality (3.4) in [2]. Consequently, inequality (4.2) is an extension of (3.4) in [2].

REFERENCES

- [1] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, UK, 1952.
- [2] Gao Mingzhe, Tan Li and Lokenath Debnath, Some improvements on Hilbert's integral inequality, *J. Math. Anal. Appl.*, Vol. **229**(1999), 682-689.
- [3] Gao Mingzhe, On the Hilbert inequality, *Zeitschrift für Analysis und ihre Anwendungen*, Vol. **18**, No. 4 (1999), 1117-1122.
- [4] Leping He, Mingzhe Gao and Weijian Jia, On the improvement of the Hardy-Hilbert's integral inequality with parameters, *Journal of Inequalities in Pure and Applied Mathematics*, Vol. **4**(2003), No. 5, Art. 94, [ONLINE: http://jipam.vu.edu.au/v4n5/108_03.html].
- [5] Yang Bicheng, On Hardy-Hilbert's integral inequality, *J. Math. Anal. Appl.*, Vol. **261**(2001), 295-306.
- [6] He Leping, Gao Mingzhe and Wei Shangrong, A note on Hilbert's inequality, *Mathematical Inequalities & Applications*, Vol. **6**(2003), No. 2, 283-288.
- [7] Gao Mingzhe, Wei Shangrong and He Leping, On the Hilbert inequality with weights, *Zeitschrift für Analysis und ihre Anwendungen*, Vol. **21**(2002), No. 1, 257-263.
- [8] D.V. Widder, An inequality related to one of Hilbert's, *J. London Math. Soc.* Vol. **4**(1924), 194-198.

(He Leping) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, JISHOU UNIVERSITY, JISHOU HUNAN 416000, PEOPLE'S REPUBLIC OF CHINA
E-mail address: lepinghe65729@163.com

(S.S. Dragomir) SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO BOX 14428, MELBOURNE CITY M.C., VICTORIA 8001, AUSTRALIA.
E-mail address: sever.dragomir@vu.edu.au
URL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>

(Yang Qiaoshun) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, NORMAL COLLEGE, JISHOU UNIVERSITY, JISHOU, HUNAN, 416000, PEOPLE'S REPUBLIC OF CHINA
E-mail address: yqs244@163.com